

Riemannian groupoids and solitons for three-dimensional homogeneous Ricci and cross curvature flows

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1 Introduction

In recent years, there has been significant progress towards understanding geometric flows of Riemannian metrics, most notably with Hamilton's and, later,

Perelman's work on Ricci flow (See, e.g., [Ham-83], [Ham-95a], [P-02], [P-03], [KL-06], [CZ-06], [MT-06]). An important method in the analysis of the Ricci flow is a careful classification and analysis of singularities of solutions to the flow [Ham-95a], often using a compactness theorem about manifolds with some kind of curvature bound (e.g., [Ham-95b], [Lu-01], [P-02], [G-03], [Lo-05]). By using a compactness theorem, one may extract a limit of solutions as the flow approaches a singularity, and the limit gives information about the asymptotic behavior of the flow.

The limit of a parabolic flow is expected to be highly symmetric, usually some type of self-similar solution, also called a soliton. A soliton is a generalization of a fixed point of a flow; in fact, it is a solution which is a fixed point except that the metric could be changing by time dependent diffeomorphisms and rescaling.

In this paper, we study three-dimensional homogeneous geometries. These geometries are easier to study because we are able to describe the metric explicitly and also exhibit a large number of diffeomorphisms to find limit soliton metrics. In general, exhibiting these diffeomorphisms is likely to be much more difficult. Three-dimensional homogeneous solutions of Ricci flow were first studied by Isenberg and Jackson [IJ-92] and later by Knopf-McLeod [KM-01]. The solutions of the simply connected homogeneous solutions were described in some detail. These solutions are quite interesting since they mostly exhibit a particular singularity type (Type III) and are often collapsing with bounded curvature. Later, Lott [Lo-05] was able to use the formalism of Riemannian groupoids to better understand the case of compact homogenous geometries and gave a complete classification in dimension 3.

The purpose of this paper is to apply the techniques of Riemannian groupoids to study the long term behavior of solutions of the negative cross curvature flow (XCF), a geometric flow on three-manifolds first introduced by Chow and Hamilton [CH-04]. The behavior of the simply connected geometries was first given by Cao, Ni, and Saloff-Coste [CNS-07]. We explain what happens to compact quotients of homogeneous solutions to XCF in a way similar to Lott's work on Ricci flow. We include detailed analysis of the Ricci flow situation as well, both to better explain our coordinates, a few of which differ from Lott's treatment, and to emphasize the similarities in the techniques and utility of the Riemannian groupoid formalism.

The rest of the paper is organized as follows. First we review the notions of Riemannian groupoids. We then review relevant aspects of the Ricci flow and cross curvature flow, together with theory of singularities and soliton solutions. We then give detailed descriptions of the homogeneous geometries Nil, Sol, $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, and $\widetilde{\mathrm{Isom}}(\mathbb{H}^2)$. Note that we choose not to include the other homogeneous geometries since they lack the complexity of these; that is, there is no need to consider changing diffeomorphisms in their convergence.

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2 Riemannian Groupoids

Haefliger first introduced the notion of Riemannian groupoid [?]. We will primarily follow the exposition in [Lo-05]. In order to emphasize pieces, we will re-introduce the definitions. We make an effort to provide the minimal number of definitions to understand the statement of convergence. A Riemannian groupoid is a structure that encapsulates the notions of a manifold, orbifold, and quotient manifold in the same global definition. Two excellent references for smooth groupoids are the books [MM-03] and [Ma-05] (both sources refer to the smooth groupoids used here as Lie groupoids). Riemannian groupoids were previously introduced in [GGHR-89], but the definition is slightly different (in Lott's treatment, one only needs a Riemannian metric on $G^{(0)}$ and not on $G^{(1)}$).

Before we recall the definition, let us give two examples which give the flavor of some things that a groupoid can do.

Example 1 (Manifold with charts) Let $\{U_i\}_{i \in I}$ be an open covering of a Riemannian manifold M . The groupoid perspective represents M as two pieces, $G^{(0)} = \coprod_{i \in I} U_i$, where \coprod denotes the disjoint union, and $G^{(1)}$, which consists

of maps between the two points in the disjoint union which correspond to the same point in the covering (i.e., if $x \in U_i \cap U_j$, there is a map $(x_i \rightarrow x_j) \in G^{(1)}$ mapping the corresponding points in the disjoint union). Note that there is always the identity map which maps a point to itself, which we may consider as an embedding $e : G^{(0)} \rightarrow G^{(1)}$. Every element in $G^{(1)}$ looks like $(x \rightarrow y)$ where $x, y \in G^{(0)}$, so there are source and range maps $s : G^{(1)} \rightarrow G^{(0)}$, $r : G^{(1)} \rightarrow G^{(0)}$ that look like $s(x \rightarrow y) = x$, $r(x \rightarrow y) = y$. Furthermore, if the source of $\gamma_1 \in G^{(1)}$ is equal to the range of $\gamma_2 \in G^{(1)}$, e.g., $\gamma_1 = (y \rightarrow z)$ and $\gamma_2 = (x \rightarrow y)$, then there is a product $\gamma_1 \gamma_2$ which essentially is associativity, e.g.

$$\begin{aligned} \gamma_1 \gamma_2 &= (y \rightarrow z) (x \rightarrow y) = x \rightarrow y \rightarrow z \\ &= x \rightarrow z. \end{aligned}$$

Note that this only works because the y is the same point, otherwise this product is not defined. Furthermore, there are inverses, $(x \rightarrow y)^{-1} = (y \rightarrow x)$.

Example 2 (Quotient by a group action) Let Γ be a group acting on a space X from the right. We will consider a groupoid structure that represents the quotient space X/Γ . Here we let $G^{(0)} = X$ and $G^{(1)} = \bigcup_{\gamma \in \Gamma, x \in G^{(0)}} \{(x \rightarrow x\gamma)\}$.

It is easy to see that the maps e, s, r are defined here in much the same way as Example 1. The group action hypotheses imply that the product is well defined.

Let us recall the general definition of a groupoid:

Definition 1 A groupoid $G = (G^{(0)}, G^{(1)}, e, s, r, \cdot)$ is a 6-tuple such that

1. $G^{(0)}$ and $G^{(1)}$ are sets.
2. The unit map $e : G^{(0)} \rightarrow G^{(1)}$ is an injection.
3. The source and range maps $s, r : G^{(1)} \rightarrow G^{(0)}$ satisfy $s \circ e = r \circ e$ are the identity map.
4. The partially defined multiplication $\cdot : G^{(1)} \times G^{(1)} \rightarrow G^{(1)}$, usually denoted by juxtaposition, satisfies the following:
 - (a) If $\gamma, \gamma' \in G^{(1)}$, then the product $\gamma\gamma'$ is defined only if $s(\gamma) = r(\gamma')$; in this case, $s(\gamma\gamma') = s(\gamma')$ and $r(\gamma\gamma') = r(\gamma)$.
 - (b) The product is associative, i.e., $(\gamma\gamma')\gamma'' = \gamma(\gamma'\gamma'')$, if both sides make sense.
 - (c) $\gamma e(s(\gamma)) = e(r(\gamma))\gamma = \gamma$.
 - (d) For any $\gamma \in G^{(1)}$, there is an element $\gamma^{-1} \in G^{(1)}$ such that $s(\gamma^{-1}) = r(\gamma)$, $r(\gamma^{-1}) = s(\gamma)$, $\gamma\gamma^{-1} = e(r(\gamma))$, and $\gamma^{-1}\gamma = e(s(\gamma))$.

Remark 2 Elements of $G^{(0)}$ are called objects and elements of $G^{(1)}$ are called arrows.

Definition 3 A trivial groupoid is one in which $G^{(1)} = G^{(0)}$ and s and r are both the identity map.

Note that in Definition 1, if one considers $G^{(1)}$ to consist of maps of singletons $(x \rightarrow y)$, then each of the axioms make quite a bit of sense: the unit is $e(x) = (x \rightarrow x)$, the source and range maps are $s(x \rightarrow y) = x$ and $r(x \rightarrow y) = y$, associativity ensures composition is okay, and inversion is $(x \rightarrow y)^{-1} = (y \rightarrow x)$.

Remark 4 It might be tempting to replace elements γ in $G^{(1)}$ by elements $(s(\gamma), r(\gamma))$ in $G^{(0)} \times G^{(0)}$. However, often there will be more than one element of $G^{(1)}$ corresponding to $(s(\gamma), r(\gamma))$. See, e.g., Example 5.

The actual space represented by a groupoid is the orbit space, defined now. We essentially want the space to be $G^{(0)}$ modulo the identifications made in $G^{(1)}$.

Definition 5 The orbit O_x of a point $x \in G^{(0)}$ is defined to be

$$O_x = s(r^{-1}(x)).$$

Note that this means that the orbit consists of all points which map to x via an arrow in $G^{(1)}$. The quotient space $G^{(0)} / \sim$, where $x \sim y$ if and only if $y \in O_x$, is called the orbit space.

Definition 6 A pointed groupoid (G, O_x) is a groupoid G together with a distinguished orbit O_x .

Often the orbit space is the actual space we are interested in. In Example 1 we see that M is the orbit space, and in Example 2 we see that X/Γ is the orbit space.

We want a notion which essentially tells us if the orbit spaces of two groupoids are the same. For instance, we would like to know that the trivial groupoid where $G^{(0)} = M$ and $G^{(1)} = M$ is equivalent to Example 1. The first guess might be to define isomorphisms in the categorical sense as appropriate morphisms between groupoids with inverses. This turns out to be too strong a requirement, so we introduce a weaker form of equivalence. We begin with the notion of localization, which is essentially the same procedure that we used to construct Example 1. The idea is that we can always take a groupoid and turn it into a disjoint union of open sets which get identified via $G^{(1)}$.

Definition 7 Let $U = \{U_i\}_{i \in I}$ be a cover of $G^{(0)}$. The localization of a groupoid G is the groupoid G_U given by

$$G_U^{(0)} = \coprod_{i \in I} U_i = \bigcup_{i \in I, x \in U_i} (i, x)$$

and

$$G_U^{(1)} = \bigcup_{i, j \in I, \gamma \in s^{-1}(U_i) \cap r^{-1}(U_j)} (i, \gamma, j).$$

The unit map is $e(i, x) = (i, e(x), i)$. The source and range maps are $s(i, \gamma, j) = (i, s(\gamma))$, $r(i, \gamma, j) = (j, r(\gamma))$. The product is $(i, \gamma, j)(j, \gamma', k) = (i, \gamma\gamma', k)$.

We have the following definition of equivalence.

Definition 8 Two groupoids G and G' are equivalent if each has a localization G_U and $G'_{U'}$ such that G_U is isomorphic to $G'_{U'}$.

Note that the property of being equivalent is weaker than the property of being isomorphic. It will be important to differentiate between equivalence and isomorphism, since the groupoid structure encodes more than the equivalence class. In particular, we may consider Ricci flow on trivial groupoids representing compact manifolds. The limit may not be a manifold, and hence it is not a trivial groupoid. However, if we consider equivalent groupoids, there may be a groupoid limit.

Example 3 (Localization of trivial groupoid) We see that Example 1 is a localization of the trivial groupoid, so they are equivalent. Note that if a Riemannian manifold has a uniform upper bound on sectional curvature bound, then one can take geodesic balls of a uniform size as the coordinate patches, as was exploited in [Fu-88] and [G-03].

Example 4 (Localization of a quotient) If the quotient is a manifold, we see that Example 2 is equivalent to the trivial groupoid on the quotient (or orbit space) by taking disjoint copies of the same regularly covered neighborhoods.

Smoothness of a groupoid will allow us to consider the maps in $G^{(1)}$ as smooth diffeomorphisms on some small open sets. In essence, this makes the maps in $G^{(1)}$ into germs of diffeomorphisms of $G^{(0)}$. The formal definition is the following.

Definition 9 *A groupoid G is smooth if*

1. $G^{(0)}$ and $G^{(1)}$ are smooth manifolds (but only assume that $G^{(0)}$ is Hausdorff and second countable),
2. e is a smooth embedding,
3. r and s are smooth submersions, and
4. multiplication is a smooth map from $\{(\gamma, \gamma') \in G^{(1)} \times G^{(1)} : s(\gamma) = r(\gamma')\}$ to $G^{(1)}$ and inversion is a smooth map.

Remark 10 *Based on the definition of groupoid, that r is a smooth submersion follows from s being a smooth submersion, but we include both in the definition to make it look more symmetric.*

Definition 11 *We refer to the dimension of the groupoid as the dimension of $G^{(0)}$.*

Definition 12 *If r and s are local diffeomorphisms, then the groupoid is said to be étale. (Note: a map is said to be étale if it is a local diffeomorphism.)*

Remark 13 *Often we will deal with groupoids which are not naturally étale. The groupoid can often be made étale by putting the sheaf topology on $G^{(1)}$, but in general we will not find a need to make our groupoids étale.*

In Examples 1 and 2, we see that the arrows come from local diffeomorphisms. This can be made precise with the following definition.

Definition 14 *A local bisection is a smooth map $\sigma : U \rightarrow G^{(1)}$, where $U \subset G^{(0)}$ is open, such that $s \circ \sigma$ is the identity and the map $r \circ \sigma : U \rightarrow r \circ \sigma(U)$ is a diffeomorphism. We use $\mathcal{B}^{loc}(G)$ to refer to the local bisections and $\mathcal{D}^{loc}(G) = \{\phi = r \circ \sigma : \sigma \in \mathcal{B}^{loc}(G)\}$ to refer to the local diffeomorphisms they generate.*

It is not hard to see that given any element $\gamma \in G^{(1)}$, there is a local bisection σ with $\gamma = \sigma(s(\gamma))$ (see [MM-03, Prop 5.3] or [Ma-05, Prop 1.4.9]). The idea is that the property that r and s are submersions is equivalent to the statement that for any $\gamma \in G^{(1)}$ there is an open set U containing γ such that $\{(s(\gamma'), r(\gamma')) : \gamma' \in U\}$ is the graph of a diffeomorphism. This is because we

may take a local section of s which is transverse to the fibers of r , giving the graph of a diffeomorphism. Thus we may think of $G^{(1)}$ as containing germs of diffeomorphisms. If the groupoid is étale, then this diffeomorphism is unique up to shrinking the domain and range.

Example 5 (Jets of local diffeomorphisms) Given a manifold M , we may define groupoids of jets of local diffeomorphisms of M , denoted $J_k = J_k(M)$, as follows. For each k , we define $J_k^{(0)} = M$. We can define $J_k^{(1)}$ as pointed diffeomorphisms $\phi : (U, p) \rightarrow (V, q)$ for open neighborhoods U of p and V of q modulo an equivalence relation. For $k = 0$, two maps $\phi : (U, p) \rightarrow (V, q)$ and $\phi' : (U', p') \rightarrow (V', q')$ are equivalent if $p = p'$ and $q = q'$. For arbitrary k , two maps are equivalent if $p = p'$, $q = q'$, and all derivatives at p of order less than or equal to k are equal. The source and range maps are defined as $s(\phi) = p$ and $r(\phi) = \phi(p) = q$.

The jet groupoids are not naturally étale, though they can be made étale by choosing the sheaf topology. We will generally not do this. Also, given a Riemannian metric on M , there is a natural Riemannian metric on the $J_k^{(1)}$ defined using the Riemannian metric and the Riemannian connection. This will be important later, where we use Hausdorff convergence of closed subsets of $J_k^{(1)}$ to define convergence of Riemannian groupoids.

Given a diffeomorphism $F : M \rightarrow M'$, there is a map $J_k^{(1)}(M)$ to $J_k^{(1)}(M')$ given by taking $[\gamma] \in J_k^{(1)}(M)$ to $[F \circ \gamma \circ F^{-1}] \in J_k^{(1)}(M')$. We denote $[F \circ \gamma \circ F^{-1}]$ by $F_*\gamma$.

Note that each self-diffeomorphism $F : M \rightarrow M$ induces a global bisection of each jet groupoid.

A smooth groupoid can be given a Riemannian structure by a putting Riemannian metric on $G^{(0)}$ that respects the maps in $G^{(1)}$ (i.e., these maps act as isometries).

Definition 15 *A smooth groupoid G is Riemannian if there is a Riemannian metric g on $G^{(0)}$ so that elements of $\mathcal{D}^{loc}(G)$ act as Riemannian isometries.*

There is a natural distance structure on the orbit space of a Riemannian groupoid (actually, it is only a pseudo-distance, since distinct points may have zero distance between them).

Definition 16 *A smooth path α in G is a partition $0 = t_0 \leq t_1 \leq \dots \leq t_k = 1$ and a sequence*

$$\alpha = (\gamma_0, \alpha_1, \gamma_1, \alpha_2, \dots, \alpha_k, \gamma_k)$$

where $\alpha_i : [t_{i-1}, t_i] \rightarrow G^{(0)}$ is a smooth path and $\gamma_i \in G^{(1)}$ with $\alpha_i(t_{i-1}) = r(\gamma_{i-1})$ and $\alpha_i(t_i) = s(\gamma_i)$. The length of a path is given by

$$L(\alpha) = \sum_{i=1}^k L(\alpha_i).$$

Definition 17 The pseudometric d on the orbit space of a Riemannian groupoid is given by

$$d(O_x, O_y) = \inf \{L(\alpha)\}$$

where the infimum is taken over all smooth paths with $s(\gamma_0) = x$ and $r(\gamma_k) = y$. If d is a metric and the orbits are closed, we say the groupoid is closed.

Remark 18 Lott [Lo-05] points out that Haefliger [?] and Salem [Sa-88] show how to form a closed groupoid by embedding the groupoid into the jet groupoid J_1 and taking the closure. In general, the topology on the space of groupoids will not see the difference between a groupoid and its closure, much like the Gromov-Hausdorff distance does not see a difference between a metric space and its completion.

Definition 19 We may define the metric balls $B_R(O_x) \subset G^{(0)}$ as the union of all orbits which are a distance less than R away from the orbit O_x .

Given these definitions, we could define Gromov-Hausdorff distance of Riemannian groupoids. Instead, we only define C^k convergence. The idea is that we must have local convergence of the Riemannian metrics on $G^{(0)}$ and we must also have local convergence of the arrows $G^{(1)}$.

Definition 20 (Convergence of Riemannian groupoids) Let $\{(G_i, g_i, O_{x_i})\}_{i=1}^\infty$ be a sequence of closed, pointed, n -dimensional Riemannian groupoids and let $(G_\infty, g_\infty, O_{x_\infty})$ be a closed, pointed Riemannian groupoid. Let J_k be the groupoid of k -jets of local diffeomorphisms of $G_\infty^{(0)}$. Then we say that $\lim_{i \rightarrow \infty} (G_i, O_{x_i}) = (G_\infty, O_{x_\infty})$ in the pointed C^k topology if for all $R > 0$,

1. There exists $I = I(R)$ such that for all $i \geq I$ there are pointed diffeomorphisms

$$\phi_{i,R} : B_R(O_{x_\infty}) \rightarrow B_R(O_{x_i})$$

so that

$$\lim_{i \rightarrow \infty} \phi_{i,R}^* g_i|_{B_R(O_{x_i})} = g_\infty|_{B_R(O_{x_\infty})}$$

in $C^k(B_R(O_{x_\infty}))$.

2. The sets

$$\phi_{i,R}^* [s_i^{-1}(B_{R/2}(O_{x_i})) \cap r_i^{-1}(B_R(O_{x_i}))]$$

(see Example 5) converge to $s_\infty^{-1}(B_{R/2}(O_{x_i})) \cap r_\infty^{-1}(B_R(O_{x_i}))$ in the Hausdorff metric on $J_k^{(1)}(G_\infty^{(0)})$.

Remark 21 As noted by Lott [Lo-05], for $k \geq 1$, we need only consider the convergence in the space of 1-jets, since the maps are local isometries and they are entirely determined by their 1-jets. We keep the k in the definition here for symmetry in the definition.

Remark 22 In [Fu-88] and [G-03], instead of convergence in the space of jets, convergence in the space of continuous maps is considered. We note that if all of the arrows can be extended to smooth maps from a fixed domain (such as a Euclidean ball), then the Arzela-Ascoli theorem tells us that convergence of the jets implies convergence in C^k of the maps. In the examples in the rest of the paper, the arrows will come from globally defined maps, and we will therefore deal only with these maps without reference to jets.

In this paper, we will primarily prove C^0 convergence. Although it is not difficult to prove convergence in C^∞ , we restrict to C^0 for clarity of exposition. In all of our examples, the convergence will be explicit and straightforward.

Groupoid convergence allows one to see collapsing in the following sense.

Definition 23 If we start with a sequence of Riemannian groupoids $\{G_i\}_{i=1}^\infty$ whose orbits are discrete and they converge to a limit groupoid G_∞ such that the orbit space is not discrete, we say that the sequence collapses.

3 Solitons on Ricci and Cross Curvature Flows

Although many of the ideas here apply to higher dimensions (some examples are given by Lott [Lo-05]), we restrict ourselves to dimension 3. In the sequel, let (M, g) be a three-dimensional Riemannian manifold.

3.1 Introduction to RF and XCF

The Ricci flow was first introduced by Hamilton [Ham-83] to study three-dimensional Riemannian manifolds. The Ricci flow is a solution to the partial differential equation on Riemannian metrics given by

$$\frac{\partial}{\partial t} g = -2 \operatorname{Rc}(g). \quad (\text{RF})$$

It is well known that the Ricci flow is weakly parabolic and has a unique solution for short time (see [Ham-83]). For future use, we note that the Ricci tensor is invariant under rescaling of the metric, i.e., for any positive constant c ,

$$\operatorname{Rc}(cg) = \operatorname{Rc}(g). \quad (1)$$

The cross curvature flow on a three-dimensional manifold was first proposed by Chow and Hamilton in [CH-04]. Define the tensor P^{ij} as

$$\begin{aligned} P^{ij} &= R^{ij} - \frac{1}{2} R g^{ij} \\ &= g^{ik} g^{j\ell} R_{k\ell} - \frac{1}{2} R g^{ij}. \end{aligned}$$

Since we are in dimension 3, we can diagonalize the Ricci tensor with an orthonormal frame $\{e_1, e_2, e_3\}$ and make P diagonal with $P(\omega^i, \omega^i)$ equal to the

sectional curvature $K(e_j \wedge e_k)$, where $\{\omega^1, \omega^2, \omega^3\}$ is the dual coframe and $\{i, j, k\}$ are distinct.

Let V_{ij} be the inverse of P^{ij} (if it exists) and then we define the cross curvature tensor as

$$h_{ij} = \left(\frac{\det P^{ij}}{\det g^{ij}} \right) V_{ij}.$$

Notice that $V_{ij} = \frac{1}{\det P^{ij}} \text{adj}(P)$ so the cross curvature tensor exists even if P is not invertible (though it may not be an elliptic operator of g). We note the following scaling property of the cross curvature tensor

$$h(cg) = \frac{1}{c} h(g) \quad (2)$$

for a positive constant c .

Definition 24 *The (negative/positive) cross curvature flow (\pm XCF) is the flow of Riemannian metrics solving*

$$\frac{\partial}{\partial t} g = -2h(g) \quad (-\text{XCF})$$

or

$$\frac{\partial}{\partial t} g = 2h(g). \quad (+\text{XCF})$$

When we omit the $+$ or $-$, we are referring to either flow.

Because our singularity models may change the direction of the flow, it will often become irrelevant which direction we are considering. However, the direction is very important for existence results. It was shown that $+\text{XCF}$ exists if the sectional curvature is positive and $-\text{XCF}$ exists if the sectional curvature is negative [Buc-06]. In other cases, the equation makes sense, but there may not be a unique solution flow. In the cases of homogeneous spaces, the partial differential equation reduces to an ordinary differential equation and thus has a unique solution for a short time.

3.2 Solitons

In this section we review soliton techniques for geometric flows. Consider any geometric flow given by

$$\frac{\partial g}{\partial t} = -2v(g) \quad (3)$$

where v is a symmetric two tensor which is function of the metric (e.g., Rc , $\pm h$). Furthermore, suppose that for any positive constant c ,

$$v(cg) = c^p v(g) \quad (4)$$

for some integer p and v is natural, i.e.,

$$v(\phi^* g) = \phi^*(v(g))$$

for any diffeomorphism $\phi : M \rightarrow M$. If $v = \text{Rc}$ then $p = 0$ by (1) and if $v = \pm h$ then $p = -1$ by (2).

Definition 25 A self-similar solution is a solution of the form

$$g(t) = \sigma(t) \phi_t^* g_0$$

where σ is a positive function with $\sigma(0) = 1$, ϕ_t is a one-parameter family of diffeomorphisms of M with ϕ_0 the identity, and g_0 is a fixed Riemannian metric on M .

Definition 26 A soliton is a metric g_0 such that there exists a vector field X on M and a constant α such that

$$-2v_0 = L_X g_0 + \alpha g_0.$$

The soliton is said to be a steady soliton if $\alpha = 0$.

We give special names to solitons for Ricci flow and cross curvature flow.

Definition 27 We refer to solitons of the Ricci flow as Ricci solitons and solitons of the cross curvature flow as XC solitons. Note that XC solitons are solitons for both positive and negative cross curvature flows.

Remark 28 If $p \neq 1$ then the metric can be rescaled to produce a soliton with $\alpha \in \{-1, 0, 1\}$.

The following two propositions are the obvious generalizations of [CK-04, Lemma 2.4 on p. 23] and [CCG+07, Proposition 1.3 and its successive remarks on pp. 3-4].

Proposition 29 (Self-similar iff soliton) If $g(t)$ is a self-similar solution for $t \in [0, T)$ then $g(0)$ is a soliton. Conversely, if g_0 is a soliton, then there exists $T > 0$ and a self-similar solution $g(t)$ for $t \in [0, T)$ with $g(0) = g_0$.

Proof. If $g(t)$ is a self-similar solution, then

$$g(t) = \sigma(t) \phi_t^* g_0.$$

Differentiating with respect to t , we get

$$-2v(\sigma(t) \phi_t^* g_0) = \frac{d\sigma}{dt}(t) \phi_t^* g_0 + \sigma(t) \phi_t^* (L_X g_0),$$

where X is the solution to $X(\phi_t(p)) = \frac{d}{dt} \phi_t(p)$ for all $p \in M$. By (4),

$$v(\sigma(t) \phi_t^* g_0) = (\sigma(t))^p \phi_t^* v(g_0)$$

we can drop the pullbacks and get

$$-2(\sigma(t))^p v(g_0) = \frac{d\sigma}{dt}(t) g_0 + L_{\sigma(t)X} g_0. \quad (5)$$

At $t = 0$, this is exactly

$$-2v(g_0) = \frac{d\sigma}{dt}(0) g_0 + L_{X(0)} g_0.$$

Conversely, if $g(0)$ is a soliton, then

$$-2v(g_0) = L_X g_0 + \alpha g_0.$$

Let $\sigma(t) = (1 + (1-p)\alpha t)^{1/(1-p)}$ if $p \neq 1$ and $\sigma(t) = \exp(\alpha t)$ if $p = 1$ (so $\frac{d\sigma}{dt} = \alpha \sigma^p$). Let ϕ_t be the diffeomorphisms generated by $(\sigma(t))^{p-1} X$. Then the metric $g(t) = \sigma(t) \phi_t^* g_0$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= \frac{d\sigma}{dt}(t) \phi_t^* g_0 + \phi_t^* (L_{(\sigma(t))^p X} g_0) \\ &= (\sigma(t))^p \phi_t^* (\alpha g_0 + L_X g_0) \\ &= -2(\sigma(t))^p \phi_t^* (v(g_0)) \\ &= -2v(\sigma(t) \phi_t^* g_0). \end{aligned}$$

■

For the rest of this paper, assume for simplicity that $p \neq 1$. We will primarily be concerned with $p = 0$ for Ricci flow and $p = -1$ for cross curvature flow. There will always be a corresponding expression if $p = 1$ which we will not provide.

Because of this proposition, we will often interchange the two terms. Note that we must assume the existence of a solution.

Proposition 30 (Canonical form) *Suppose $g(t)$ is a self similar solution. Then there exist diffeomorphisms ψ_t and a constant $\alpha \in \mathbb{R}$ such that*

$$g(t) = (1 + (1-p)\alpha t)^{1/(1-p)} \psi_t^* g_0.$$

Proof. We have supposed that

$$g(t) = \sigma(t) \phi_t^* g_0.$$

By (5), we have

$$\begin{aligned} -2(\sigma(t))^p v(g_0) &= \frac{d\sigma}{dt}(t) g_0 + L_{\sigma(t)X} g_0, \\ -2v(g_0) &= \sigma(t)^{-p} \frac{d\sigma}{dt}(t) g_0 + L_{\sigma(t)^{1-p} X} g_0 \\ &= \frac{1}{1-p} \frac{d\sigma^{1-p}}{dt} g_0 + L_{\sigma^{1-p} X} g_0. \end{aligned}$$

Differentiating this equation again with respect to t give us

$$0 = \frac{1}{1-p} \frac{d^2 \sigma^{1-p}}{d^2 t} g_0 + L_{\tilde{X}} g_0,$$

where $\tilde{X}(t) = \frac{d\sigma^{1-p}}{dt} X + \sigma^{1-p} \frac{dX}{dt}$. So either $\frac{d^2 \sigma^{1-p}}{d^2 t} = 0$ or $g_0 = L_Y g_0$ with $Y = -\tilde{X} / \left(\frac{1}{1-p} \frac{d^2 \sigma^{1-p}}{d^2 t} \right)$. In the first case,

$$\sigma(t) = (1 + (1-p)\alpha t)^{1/(1-p)},$$

since $g(0) = g_0$. In the second case,

$$-2v(g_0) = L_{\beta X + \gamma Y} g_0,$$

where $\beta = \sigma^{1-p}$ and $\gamma = \left(\frac{1}{1-p} \frac{d\sigma^{1-p}}{dt} \right)$ and so we may choose $\alpha = 0$. The proof is completed by Proposition 29. ■

Corollary 31 *All Ricci solitons can be put in the form*

$$g(t) = (1 + \alpha t) \psi_t^* g_0,$$

and all XC solitons can be put in the form

$$g(t) = (1 + 2\alpha t)^{1/2} \psi_t^* g_0.$$

Remark 32 *Following Lott [Lo-05], we will base our self-similar solutions at $g_1 = g(1)$ instead of $g_0 = g(0)$. In this case, the canonical forms are $g(t) = \alpha t \psi_t^* g_1$ for Ricci flow and $g(t) = (2\alpha t)^{1/2} \psi_t^* g_1$ for cross curvature flow.*

3.3 Theory of singularities

In order to understand the geometry of a limit solution, one must look at the appropriate length scale. For instance, given any Riemannian manifold (\mathcal{M}, g) and a point $p \in \mathcal{M}$, one could consider the manifold gotten by the limit of (\mathcal{M}^n, sg) where $s \rightarrow \infty$. Since the space is a Riemannian manifold, this will converge in the Gromov-Hausdorff sense to Euclidean space $(\mathbb{R}^n, g_{\mathbb{E}})$. On the other hand, if one takes the unit sphere metric $(S^n, g_{S^n(1)})$ and looks at the limit $(S^n, sg_{S^n(1)})$ where $s \rightarrow 0$, it is clear that the sectional curvatures go to infinity. As we are taking limits, in order to understand the geometry of a particular solution, we will wish to rescale in such a way that we get a reasonable limit that has, if possible, nonzero curvatures. This is what we will call the *geometric limit*. The general process for rescaling is to rescale so that the maximum sectional curvature (in absolute value) does not go to zero or infinity. If $g(t)$ is a solution to the geometric equation (3), we will do a parabolic rescaling so that the limit is also a solution to the flow. Suppose $g(t)$ is defined on a maximal time interval $[0, T)$. The usual rescaling as the flow goes to the singularity at $T \in (0, \infty]$ is

as follows (see [Ham-95a], [CK-04] [CLN-06]). We take an increasing sequence t_i converging to T (or going to infinity if $T = \infty$) and consider the sequence of metrics

$$g_i(t) = M(t_i) g\left(\frac{t}{M(t_i)} + t_i\right).$$

These solutions have the property that $g_i(0) = g(t_i)$ for some function $M(t)$. Note that if we take

$$M(t) = \sup_{\mathcal{M}} |\text{Rm}(g(t))| \quad (6)$$

then $\sup_{\mathcal{M}} |\text{Rm}(g_i(0))| = 1$.

In following Lott [Lo-05], we consider a continuous deformation (so instead of taking a sequence t_i , we take a parameter s) with base metric $g_s(1)$ (instead of $g_i(0)$). If $T = \infty$, we will consider deformations of the form

$$g_s(t) = f(s) g\left(\frac{t-1}{(f(s))^{1-p}} + s\right), \quad (7)$$

where p is defined by (4). These have the property that $g_s(1) = g(s)$ and that $g_s(t)$ is defined for t in

$$[1 - (f(s))^{1-p} s, \infty).$$

We also see that $g_s(t)$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t} g_s &= -2(f(s))^p v\left(g\left(\frac{t-1}{(f(s))^{1-p}} + s\right)\right) \\ &= -2v(g_s). \end{aligned}$$

We will then consider limits as $s \rightarrow \infty$,

$$g_\infty(t) = \lim_{s \rightarrow \infty} \phi_s^* g_s(t)$$

where ϕ_s are appropriately chosen diffeomorphisms. Note that if $f(s) = s^{-1/(1-p)}$ then $g_s(t) = s^{-1/(1-p)} g(st)$, and

$$g_\infty(t) = \lim_{s \rightarrow \infty} s^{-1/(1-p)} \phi_s^* g(st),$$

which will be a common rescaling (note that for Ricci flow we have $-1/(1-p) = -1$ and for cross curvature flow we have $-1/(1-p) = -1/2$). In this case, the limit will be defined for t in $[0, \infty)$.

In the case of $T < \infty$, we will instead look at limits defined by

$$g_s(t) = f(s) g\left(T - \left(\frac{t-1}{(f(s))^{1-p}} + s\right)\right). \quad (8)$$

These have the property that $g_s(1) = g(T-s)$, $g_s(t)$ is defined for t in the interval

$$(1 - (f(s))^{1-p} s, (f(s))^{1-p} (T - 1 - s) + 1],$$

and

$$\frac{\partial}{\partial t} g_s = 2v(g_s) \quad (9)$$

(notice that the sign is flipped). In this case, to look at the solution near the singularity, we look at the limit

$$g_T(t) = \lim_{s \rightarrow 0} \phi_s^* g_s(t)$$

for some choice of diffeomorphisms ϕ_s . Note that if $f(s) = s^{-1/(1-p)}$ then the limit is defined for t in $(0, \infty)$ and the limit looks like

$$g_T(t) = \lim_{s \rightarrow 0} s^{-1/(1-p)} \phi_s^* g_s(st).$$

It is often more important for us to understand how a particular solution of a flow compares with other solutions. In this case, we will consider certain classes of singularities. As introduced by Hamilton [Ham-95a], one can separate solutions into 4 classes of solutions (we take $M(t)$ defined by (6)):

Type I. $T < \infty$ and $\sup (T - t)^{1/(1-p)} M(t) < \infty$

Type IIa. $T < \infty$ and $\sup (T - t)^{1/(1-p)} M(t) = \infty$

Type IIb. $T = \infty$ and $\sup t^{1/(1-p)} M(t) = \infty$

Type III. $T = \infty$ and $\sup t^{1/(1-p)} M(t) < \infty$.

This singularity theory gives a canonical rescaling factor of $t^{1/(1-p)}$ designed to give limit soliton metrics based on the canonical form of soliton metrics described in Proposition 30. It is significant that this rescaling is chosen by the flow and not by the solution itself.

Remark 33 *Another canonical rescaling one might propose is one such that the volume is unchanged (often called normalized Ricci flow and normalized cross curvature flow), as used in [Ham-83], [IJ-92], [CNS-07], and others. We do not treat this particular rescaling, arguing that the geometric rescaling that keeps the curvatures bounded and the singularity rescaling are more natural in most of the cases we give here. In many of the cases we treat, rescaling so that volume is unchanged will not prevent collapsing and convergence will usually be to a collapsed flat manifold.*

For Ricci flow on three-dimensional manifolds, it is an interesting fact that most of the homogeneous solutions are Type III, with the exception of S^3 and $S^2 \times \mathbb{R}$. For negative cross curvature flow, we will actually find a Type IIb solution.

When looking at a Type III solution, we will look at the *Type III limit solution* described by the limit of *Type III rescalings*

$$s^{-1/(1-p)} g(st)$$

as $s \rightarrow \infty$. When looking at a Type I solution, we will look at the *Type I limit solution* described by the limit of *Type I rescalings*

$$s^{-1/(1-p)} g(T - st)$$

as $s \rightarrow 0$. Notice that the negative sign makes this a solution not of the original flow, but of the backward flow (9). For Type II solutions, we will need to find an appropriate geometric rescaling, since the rescalings we have proposed will take the sectional curvature to infinity.

The existence of limit solutions described in the previous paragraph is not guaranteed. By Hamilton's compactness theorem [Ham-95b], if we had a uniform lower bound on the injectivity radius of the scaled solutions, one could take this limit for Ricci flow (and it is not hard to construct a similar theorem for XCF). In the Type I Ricci flow case, Perelman showed that this bound exists [P-02]. For Type III Ricci flow this is not a reasonable assumption since collapsing does happen. Similarly, collapsing can occur for XCF. However, the idea of limit solutions can apply in the setting of Riemannian groupoids, and this is where the limits will be taken. Instead of Hamilton's theorem, Lott's compactness theorem for Ricci flow on Riemannian groupoids ([Lo-05], see also [Fu-88] and [G-03] for some of the geometric ideas) may be used to extract a limit.

4 3D homogeneous solutions

In this section we review the results on Ricci flow on three-dimensional homogeneous geometries and give the results on cross curvature flow. Solutions of the Ricci flow on three-dimensional, simply connected, homogeneous geometries were first described by Isenberg-Jackson [IJ-92] (see also [KM-01], [CK-04, Chapter 1], [CLN-06, Chapter 4, Section 7]). In general, we may start with a basis of left-invariant vector fields F_1, F_2, F_3 and consider the class of left invariant metrics such that this frame is orthogonal (but the length of the vectors in the frame is arbitrary). Solutions of the negative cross curvature flow on simply connected homogeneous geometries were first described by Cao-Ni-Saloff-Coste [CNS-07]. The homogeneous expanding solitons on Nil and Sol were described by Baird-Danielo [BD-05] and Lott [Lo-05]. The results below on Ricci flow are due to Lott [Lo-05]; in some cases we give different coordinate representations in an attempt to make the limit groupoids especially clear. The results on cross curvature flow are new.

Remark 34 *We choose to follow the convention of Lott [Lo-05] that the brackets of the frame look like $[F_i, F_j] = c_{ij}^k F_k$ where c_{ij}^k are in $\{-1, 0, 1\}$. This is different from the conventions in [KM-01] and [CNS-07], and so curvatures may look slightly different because of the discrepancy in the definition of A, B, C . We will try to point out the discrepancies in each example.*

The goal of this section is to find the limits of collapsing solutions of Ricci flow and cross curvature flow. In the process, we find Ricci and cross curvature

solitons which occur in the limit. The process is as follows. First, look at the asymptotic solutions of the flow on simply connected geometries. Since each of the following simply connected geometries is diffeomorphic to \mathbb{R}^3 , there is a wealth of diffeomorphisms available, primarily rescaling of the coordinates. We need to find diffeomorphisms so that the metrics in coordinates which are pulled back by these diffeomorphisms do not degenerate. The limit geometry may be the same, in which case the geometry admits a soliton metric, or it may be different, in which case the geometry converges to another geometry. Lastly, for a compact homogenous manifold, we consider the equivalent groupoid consisting of the universal cover together with arrows described by the action of the fundamental group as deck transformations. If the limit of the arrows describes a continuous group \mathcal{D}^{loc} , there is collapsing. We note that it is extremely important to find the right coordinates so that the diffeomorphisms may be written out explicitly.

We shall only look at four of the possible three-dimensional homogeneous geometries because we wish to emphasize the importance of the changing diffeomorphisms. One may consider the remaining geometries, but the effects of the diffeomorphisms is much more trivial than the effects in Nil, Sol, $\widetilde{\text{SL}}(2, \mathbb{R})$, and $\widetilde{\text{Isom}}(\mathbb{E}^2)$.

4.1 Nil

Recall that Nil consists of the unit upper triangular matrices,

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

and so the group multiplication is

$$(a, b, c)(x, y, z) = (x + a, y + b, z + c + ay).$$

We easily see that the following global vector fields are left invariant

$$F_1 = \frac{\partial}{\partial z}, \quad F_2 = \frac{\partial}{\partial x}, \quad F_3 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z},$$

and we easily see that

$$[F_2, F_3] = F_1$$

and all other brackets are zero. We can also define the dual forms as

$$\theta_1 = dz - xdy, \quad \theta_2 = dy, \quad \theta_3 = dx.$$

It is then clear that the following metrics are all left invariant

$$\begin{aligned} g &= A\theta_1^2 + B\theta_2^2 + C\theta_3^2 \\ &= A(dz - xdy)^2 + Bdy^2 + Cdx^2. \end{aligned} \tag{10}$$

Note that by appropriate scaling of the coordinates, we see that this is actually only a one-parameter family of metrics up to diffeomorphism. It is not difficult to see that the sectional curvatures for these metrics are

$$\begin{aligned} K(F_2 \wedge F_3) &= -\frac{3A}{4BC}, \\ K(F_3 \wedge F_1) &= \frac{A}{4BC}, \\ K(F_1 \wedge F_2) &= \frac{A}{4BC}. \end{aligned}$$

Note that this F_1 is half that used in [KM-01] and [CNS-07], so our A is $1/4$ the corresponding coefficient in those papers.

4.1.1 Ricci Flow

It is well known that the Ricci flow on the metric (10) has the form

$$\begin{aligned} \frac{dA}{dt} &= -\frac{A^2}{BC}, \\ \frac{dB}{dt} &= \frac{A}{C}, \\ \frac{dC}{dt} &= \frac{A}{B}, \end{aligned}$$

(see [IJ-92] [KM-01]). The solution is

$$\begin{aligned} A(t) &= A_0 \left(3 \frac{A_0}{B_0 C_0} t + 1 \right)^{-1/3}, \\ B(t) &= B_0 \left(3 \frac{A_0}{B_0 C_0} t + 1 \right)^{1/3}, \\ C(t) &= C_0 \left(3 \frac{A_0}{B_0 C_0} t + 1 \right)^{1/3}. \end{aligned}$$

Notice that the sectional curvatures all behave like t^{-1} , so the solution is Type III. We may pull the metric back by the diffeomorphisms

$$\phi_t(x, y, z) = \left(\frac{x}{t^{1/6}}, \frac{y}{t^{1/6}}, t^{1/6} z \right)$$

to get the metrics

$$\begin{aligned} \phi_t^* g(t) &= A_0 \left(3 \frac{A_0}{B_0 C_0} + \frac{1}{t} \right)^{-1/3} \left(dz - t^{-1/2} x dy \right)^2 + B_0 \left(3 \frac{A_0}{B_0 C_0} + \frac{1}{t} \right)^{1/3} dy^2 \\ &\quad + C_0 \left(3 \frac{A_0}{B_0 C_0} + \frac{1}{t} \right)^{1/3} dx^2. \end{aligned}$$

We note that as $t \rightarrow \infty$, this converges to the Euclidean metric. However, this is because the Ricci flow causes the metric to spread out. We may counteract this by rescaling the metric via a Type III rescaling. So, instead, we consider

$$g_s(t) = \frac{1}{s} g(st)$$

where $s \rightarrow \infty$. The idea is that $g_s(1)$ is the long term behavior of $g(t)$ after rescaling. We pull back by different diffeomorphisms

$$\psi_s(x, y, z) = \left(s^{1/3}x, s^{1/3}y, s^{2/3}z \right)$$

and get

$$\begin{aligned} \frac{1}{s} \psi_s^* g_s &= A_0 \left(3 \frac{A_0}{B_0 C_0} t + \frac{1}{s} \right)^{-1/3} (dz - xdy)^2 + B_0 \left(3 \frac{A_0}{B_0 C_0} t + \frac{1}{s} \right)^{1/3} dy^2 \\ &\quad + C_0 \left(3 \frac{A_0}{B_0 C_0} t + \frac{1}{s} \right)^{1/3} dx^2. \end{aligned}$$

As $s \rightarrow \infty$, we get the limit Ricci flow

$$\begin{aligned} g_\infty(t) &= A_0 \left(3 \frac{A_0}{B_0 C_0} t \right)^{-1/3} (dz - xdy)^2 + B_0 \left(3 \frac{A_0}{B_0 C_0} t \right)^{1/3} dy^2 \\ &\quad + C_0 \left(3 \frac{A_0}{B_0 C_0} t \right)^{1/3} dx^2. \end{aligned}$$

Notice that if we pull back by the diffeomorphism

$$\tilde{\psi}(x, y, z) = \left(C_0^{-1/2} \left(\frac{3A_0}{B_0 C_0} \right)^{-1/6} x, B_0^{-1/2} \left(\frac{3A_0}{B_0 C_0} \right)^{-1/6} y, B_0^{-1/2} C_0^{-1/2} \left(\frac{3A_0}{B_0 C_0} \right)^{-1/3} z \right)$$

we get

$$\begin{aligned} \tilde{g}_\infty(t) &= \tilde{\psi}^* g_\infty(t) \\ &= \frac{1}{t^{1/3}} (dz - xdy)^2 + t^{1/3} dy^2 + t^{1/3} dx^2. \end{aligned}$$

This is the Nil soliton from [Lo-05] and [BD-05]. We easily see that

$$\tilde{g}_\infty(t) = t \phi_t^* g$$

where

$$g = (dz - xdy)^2 + dy^2 + dx^2$$

and

$$\phi_t(x, y, z) = \left(t^{-1/3}x, t^{-1/3}y, t^{-2/3}z \right).$$

Now consider compact quotients of Nil. We can also compute the limit of isometries on $g(t)$, which are

$$\psi_s^{-1} \circ \gamma_{k,\ell,m} \circ \psi_s(x, y, z) = \left(x + s^{-1/3}k, y + s^{-1/3}\ell, z + s^{-2/3}m + s^{-1/3}ky \right).$$

If we take the limit $s \rightarrow \infty$, the the group converges to the following isometries of $g_\infty(t)$

$$\gamma_{u,v,w}(x, y, z) = (x + u, y + v, z + w + uy),$$

where u, v, w are real numbers gotten by choosing numbers $k(s), \ell(s), m(s)$ such that

$$\begin{aligned} \lim_{s \rightarrow \infty} s^{-1/3}k(s) &= u, \\ \lim_{s \rightarrow \infty} s^{-1/3}\ell(s) &= v, \\ \lim_{s \rightarrow \infty} s^{-2/3}m(s) &= w. \end{aligned}$$

Note that this means that k, ℓ, m must become large. Now, suppose we start with a compact quotient of Nil. Then the fundamental group may be represented as a subgroup of Nil acting freely, properly discontinuously. Still, it is possible for k, ℓ, m to become infinitely large, since the lattice must extend through the entirety of Nil for the quotient to be compact. Thus the lattice converges to all of the group elements, and thus the renormalized manifold converges to a groupoid whose orbit space is a point. It is interesting to note that this is, in some sense, the optimal geometry for Nil, as Gromov's almost flat manifold theorem states that any almost flat manifold must be an infranilmanifold [Gr-78] (see also [BK-81]).

4.1.2 Cross Curvature Flow

From [CNS-07], we see that the negative cross curvature flow on Nil has solutions

$$\begin{aligned} A(t) &= A_0 (1 + 7R_0^2 t)^{-1/14}, \\ B(t) &= B_0 (1 + 7R_0^2 t)^{3/14}, \\ C(t) &= C_0 (1 + 7R_0^2 t)^{3/14}, \end{aligned}$$

where $R_0 = -A_0/(2B_0C_0)$ (this is one fourth the value in [CNS-07]). Notice that all sectional curvatures behave asymptotically like $t^{-1/2}$, so there may be a XC soliton metric on Nil. We can consider

$$\begin{aligned} \frac{1}{s^{1/2}} \phi_s^* g(st) &= A_0 \frac{(1 + 7R_0^2 st)^{-1/14} s^{4/7}}{s^{1/2}} (dz - xdy)^2 + B_0 \frac{(1 + 7R_0^2 st)^{3/14} s^{2/7}}{s^{1/2}} dy^2 \\ &\quad + C_0 \frac{(1 + 7R_0^2 st)^{3/14} s^{2/7}}{s^{1/2}} dx^2 \end{aligned}$$

whose limit as $s \rightarrow \infty$ is

$$A_0 (7R_0^2 t)^{-1/14} (dz - xdy)^2 + B_0 (7R_0^2 t)^{3/14} dy^2 + C_0 (7R_0^2 t)^{3/14} dx^2,$$

where

$$\phi_s(x, y, z) = \left(s^{1/7} x, s^{1/7} y, s^{2/7} z \right).$$

Note that we may pull back by

$$\tilde{\psi}(x, y, z) = \left(C_0^{-1/2} (7R_0^2)^{-3/28} x, B_0^{-1/2} (7R_0^2)^{-3/28} y, (B_0 C_0)^{-1/2} (7R_0^2)^{-3/14} z \right)$$

to get

$$g(t) = \frac{2}{\sqrt{7} t^{1/14}} (dz - xdy)^2 + t^{3/14} (dy^2 + dx^2).$$

This is a XC soliton, since

$$g(t) = t^{1/2} \psi_t^* \left(\frac{2}{\sqrt{7}} (dz - xdy)^2 + dy^2 + dx^2 \right)$$

where

$$\psi_t(x, y, z) = \left(t^{-1/7} x, t^{-1/7} y, t^{-2/7} z \right).$$

We can also compute the limit of isometries on $g(t)$, which are

$$\phi_s^{-1} \circ \gamma_{k, \ell, m} \circ \phi_s(x, y, z) = \left(x + s^{-1/7} k, y + s^{-1/7} \ell, z + s^{-2/7} m + s^{-1/7} ky \right).$$

The group limit looks much like that for the Ricci flow, in that we get group elements

$$\gamma_{u, v, w} = (x + u, y + v, z + w + uy)$$

where u, v, w are real numbers gotten by choosing numbers $k(s), \ell(s), m(s)$ such that

$$\begin{aligned} \lim_{s \rightarrow \infty} s^{-1/7} k(s) &= u, \\ \lim_{s \rightarrow \infty} s^{-1/7} \ell(s) &= v, \\ \lim_{s \rightarrow \infty} s^{-2/7} m(s) &= w. \end{aligned}$$

Once again, we get convergence to a Riemannian groupoid whose orbit space is a point.

4.2 Sol

The group Sol is a Lie group on \mathbb{R}^3 with a group action given by

$$(a, b, c)(x, y, z) = (e^{-c} x + a, e^c y + b, z + c).$$

One can easily see that the frame

$$F_1 = e^{-z} \frac{\partial}{\partial x} + e^z \frac{\partial}{\partial y}, \quad F_2 = -\frac{\partial}{\partial z}, \quad F_3 = e^{-z} \frac{\partial}{\partial x} - e^z \frac{\partial}{\partial y}$$

is left invariant and satisfies

$$\begin{aligned} [F_1, F_2] &= -F_3, \\ [F_2, F_3] &= F_1, \\ [F_3, F_1] &= 0. \end{aligned}$$

We have a family of left invariant metrics given by

$$g = A (e^z dx + e^{-z} dy)^2 + B dz^2 + C (e^z dx - e^{-z} dy)^2.$$

This is really only a two-parameter family up to diffeomorphism, since we may rescale x and y , making A and C only well-defined up to their ratio. We may also find it useful to use alternate coordinates, which give

$$g = \left(d\tilde{z} - 2\sqrt{\frac{A}{BC}} \tilde{x} d\tilde{y} \right)^2 + d\tilde{y}^2 + \left(d\tilde{x} - 2\sqrt{\frac{C}{AB}} \tilde{z} d\tilde{y} \right)^2, \quad (11)$$

by the map described by

$$\begin{aligned} x &= e^{\frac{\tilde{y}}{\sqrt{B}}} \left(\frac{\tilde{x}}{\sqrt{C}} + \frac{\tilde{z}}{\sqrt{A}} \right), \\ y &= e^{\frac{\tilde{y}}{\sqrt{B}}} \left(-\frac{\tilde{x}}{\sqrt{C}} + \frac{\tilde{z}}{\sqrt{A}} \right), \\ z &= \frac{\tilde{y}}{\sqrt{B}} \end{aligned}$$

with inverse

$$\begin{aligned} \tilde{x} &= \frac{1}{2} \sqrt{C} (e^z x - e^{-z} y), \\ \tilde{y} &= \sqrt{B} z, \\ \tilde{z} &= \frac{1}{2} \sqrt{A} (e^z x + e^{-z} y). \end{aligned}$$

It is not hard to see that the sectional curvatures are

$$\begin{aligned} K(F_2 \wedge F_3) &= \frac{(A - C)^2 - 4A^2}{4ABC}, \\ K(F_3 \wedge F_1) &= \frac{(A + C)^2}{4ABC}, \\ K(F_1 \wedge F_2) &= \frac{(A - C)^2 - 4C^2}{4ABC}. \end{aligned}$$

An isometry $\gamma_{(a,b,c)}(x, y, z) = (a, b, c)(x, y, z)$ (expressed in the first coordinate chart) can be brought back to an action on $(\tilde{x}, \tilde{y}, \tilde{z})$ as

$$\begin{aligned} & \gamma_{(a,b,c)}(\tilde{x}, \tilde{y}, \tilde{z}) \\ &= \left(\tilde{x} + e^{\tilde{y}/\sqrt{B}+c}a - e^{-\tilde{y}/\sqrt{B}-c}b, \tilde{y} + \sqrt{B}c, \tilde{z} + e^{\tilde{y}/\sqrt{B}+c}a + e^{-\tilde{y}/\sqrt{B}-c}b \right). \end{aligned} \quad (12)$$

Note that our F_2 is one half that in [KM-01], and so our B is $1/4$ the corresponding B in that paper and [CNS-07], but agrees with [Lo-05].

4.2.1 Ricci Flow

From [KM-01] we see that

$$\begin{aligned} A, C &\sim \sqrt{A_0 C_0}, \\ B &\sim 4t, \\ A - C &\sim \frac{E}{t}. \end{aligned}$$

The sectional curvatures look like

$$\begin{aligned} K(F_2 \wedge F_3) &\sim \frac{E_1}{t}, \\ K(F_3 \wedge F_1) &\sim \frac{E_2}{t}, \\ K(F_1 \wedge F_2) &\sim \frac{E_3}{t}, \end{aligned}$$

for some constants E_1, E_2, E_3 . Thus the solution is Type III. We may take the Type III limit as

$$\begin{aligned} & \lim_{s \rightarrow \infty} s^{-1} \phi_s^* g(st) \\ &= \lim_{s \rightarrow \infty} \left[s^{-1} \sqrt{A_0 C_0} \left(e^z s^{1/2} dx + e^{-z} s^{1/2} dy \right)^2 + 4tdz^2 + s^{-1} \sqrt{A_0 C_0} \left(e^z s^{1/2} dx - e^{-z} s^{1/2} dy \right)^2 \right] \\ &= \sqrt{A_0 C_0} (e^z dx + e^{-z} dy)^2 + 4tdz^2 + \sqrt{A_0 C_0} (e^z dx - e^{-z} dy)^2 \\ &= \sqrt{A_0 C_0} (e^{2z} dx^2 + e^{-2z} dy^2) + 4tdz^2, \end{aligned}$$

where $\phi_s(x, y, z) = (s^{1/2}x, s^{1/2}y, z)$. We can pull back by

$$\psi(x, y, z) = \left((A_0 C_0)^{-1/4} x, (A_0 C_0)^{-1/4} y, z \right)$$

to get the limit soliton

$$e^{2z} dx^2 + e^{-2z} dy^2 + 4tdz^2.$$

This is the Sol soliton described in [BD-05] and [Lo-05].

We may now look at the limits of the group actions, which give

$$\begin{aligned}
& \lim_{s \rightarrow \infty} \phi_s^{-1} \circ \gamma_{(a,b,c)} \circ \phi_s(x, y, z) \\
&= \lim_{s \rightarrow \infty} \left(e^{-c}x + s^{-1/2}a, e^cy + s^{-1/2}b, z + c \right) \\
&= (e^{-c}x + u, e^cy + v, z + c)
\end{aligned}$$

if for large s , we choose a and b so that

$$\begin{aligned}
\lim_{s \rightarrow \infty} s^{-1/2}a(s) &= u, \\
\lim_{s \rightarrow \infty} s^{-1/2}b(s) &= v.
\end{aligned}$$

If we consider a discrete group which gives a compact manifold quotient, we see that we may get the limits with real u, v , though not so with c . Thus we get the group of transformations $\gamma_{(u,v,c)}$ where $u, v \in \mathbb{R}$ and $c \in \mathbb{Z}$. Clearly this limit is a groupoid whose orbit space is a circle.

4.2.2 Cross Curvature Flow

From [CNS-07], we see that the negative cross curvature flow on Sol has solutions

$$\begin{aligned}
B &\sim 2\sqrt{T_0 - t}, \\
A, C &\sim \frac{E_1}{\sqrt{T_0 - t}}, \\
A - C &\sim E_2\sqrt{T_0 - t}
\end{aligned}$$

for some positive constants E_1 and E_2 and singular time $T_0 > 1$. Note that all of the curvatures blow up like $\frac{E}{\sqrt{T_0 - t}}$, so the solution is Type I. We may now consider the Type I renormalization,

$$\begin{aligned}
& g_{T_0}(t) \\
&= \lim_{s \rightarrow 0} s^{-1/2} \phi_s^* g(T_0 - st) \\
&= \lim_{s \rightarrow 0} s^{-1/2} \left[\frac{E_1}{\sqrt{st}} \left(e^z s^{1/2} dx + e^{-z} s^{1/2} dy \right)^2 + 2\sqrt{st} dz^2 + \frac{E_1}{\sqrt{st}} \left(e^z s^{1/2} dx - e^{-z} s^{1/2} dy \right)^2 \right] \\
&= \frac{E_1}{\sqrt{t}} (e^z dx + e^{-z} dy)^2 + 2\sqrt{t} dz^2 + \frac{E_1}{\sqrt{t}} (e^z dx - e^{-z} dy)^2 \\
&= \frac{E_1}{\sqrt{t}} (e^{2z} dx^2 + e^{-2z} dy^2) + 2\sqrt{t} dz^2,
\end{aligned}$$

where

$$\phi_s(x, y, z) = (s^{1/2}x, s^{1/2}y, z).$$

Note that this equals

$$t^{1/2} \psi_t^* g(1)$$

where

$$\psi_t(x, y, z) = \left(t^{-1/2}x, t^{-1/2}y, z \right).$$

We may now look at the limit of the group actions,

$$\begin{aligned} & \lim_{s \rightarrow 0} \phi_s^{-1} \circ \gamma_{(a,b,c)} \circ \phi_s(x, y, z) \\ &= \lim_{s \rightarrow 0} \left(e^{-c}x + s^{-1/2}a, e^cy + s^{-1/2}b, z + c \right) \\ &= (e^{-c}x + u, e^cy + v, z + c) \end{aligned}$$

if

$$\begin{aligned} \lim_{s \rightarrow 0} s^{-1/2}a(s) &= u, \\ \lim_{s \rightarrow 0} s^{-1/2}b(s) &= v. \end{aligned}$$

If we consider the limit of a groupoid representing a compact manifold which is the quotient of Sol by isometries, then we see that we cannot choose $a(s)$ and $b(s)$ to be arbitrarily small (because the group must act properly discontinuously), so we must have $u = v = 0$. This corresponds to the fact that as we go to the limit, the arrows of the groupoid send elements of the ball of radius R outside the ball of radius $2R$ for any R , and so these do not survive in the limit groupoid. Thus the orbit space looks like a noncompact Sol with a quotient by the group of transformations $\gamma_{(0,0,c)}$ where $c \in \mathbb{Z}$.

4.3 $\widetilde{\text{SL}}(2, \mathbb{R})$

The homogeneous geometry $\widetilde{\text{SL}}(2, \mathbb{R})$ is diffeomorphic to \mathbb{R}^3 , but in a nontrivial way. We construct a coordinate patch on $\widetilde{\text{SL}}(2, \mathbb{R})$ following the description in [Sc-83]. $\text{PSL}(2, \mathbb{R})$ is the isometry group of the hyperbolic plane \mathbb{H}^2 . Thus we see that $\text{PSL}(2, \mathbb{R})$ acts freely and transitively on $U\mathbb{H}^2$, the unit tangent bundle of \mathbb{H}^2 , and we thus have a diffeomorphism $\text{PSL}(2, \mathbb{R}) \cong U\mathbb{H}^2$. We lift this diffeomorphism to the universal cover and easily recognize the universal cover of the right side as diffeomorphic to \mathbb{R}^3 . The lifted diffeomorphism can be described explicitly as follows. We first describe the map $\Phi : \text{PSL}(2, \mathbb{R}) \rightarrow U\mathbb{H}^2$, which is given by the action on the basepoint $(i, [0]) = (0, 1, [0]) \in U\mathbb{H}^2$ (where the unit vectors are described by their angles, $[\theta] = \theta \bmod 2\pi$, with respect to the x -axis in the half-plane model):

$$\begin{aligned} \Phi \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \left(\frac{ai + b}{ci + d}, \left[\tan^{-1} \frac{2cd}{d^2 - c^2} \right] \right) \\ &= \left(\frac{ac + bd}{c^2 + d^2}, \frac{1}{c^2 + d^2}, \left[\tan^{-1} \frac{2cd}{d^2 - c^2} \right] \right). \end{aligned} \quad (13)$$

The diffeomorphism is then gotten by lifting the map

$$\widetilde{\text{SL}}(2, \mathbb{R}) \rightarrow \text{PSL}(2, \mathbb{R}) \xrightarrow{\Phi} U\mathbb{H}^2$$

to the universal cover of $U\mathbb{H}^2$. For future use, we explicitly give the inverse of Φ :

$$\Phi^{-1}(x, y, [\theta]) = \left[\frac{1}{y^{1/2}} \begin{pmatrix} x \sin \frac{\theta}{2} + y \cos \frac{\theta}{2} & x \cos \frac{\theta}{2} - y \sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \right],$$

where $\left[\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \right]$ denotes the equivalence class of the matrix up to multiplication by -1 . We can derive the group multiplication $(a, b, [\tau])(x, y, [\theta])$ in $U\mathbb{H}^2$ to be

$$\begin{aligned} (a, b, [\tau])(x, y, [\theta]) &= \Phi(\Phi^{-1}(a, b, [\tau]) \cdot \Phi^{-1}(x, y, [\theta])) \\ &= \begin{pmatrix} a + \frac{b(x \cos \tau + \frac{1}{2}(x^2 + y^2 - 1) \sin \tau)}{\sin^2 \frac{\tau}{2} [(x + \cot \frac{\tau}{2})^2 + y^2]}, \\ \frac{by}{\sin^2 \frac{\tau}{2} [(x + \cot \frac{\tau}{2})^2 + y^2]}, \\ \left[2 \tan^{-1} \left(\frac{\sin \frac{\theta}{2} (x + \cot \frac{\tau}{2}) + y \cos \frac{\theta}{2}}{\cos \frac{\theta}{2} (x + \cot \frac{\tau}{2}) - y \sin \frac{\theta}{2}} \right) \right] \end{pmatrix} \\ &= \begin{pmatrix} a + \frac{b(x \cos \tau + \frac{1}{2}(x^2 + y^2 - 1) \sin \tau)}{\sin^2 \frac{\tau}{2} [(x + \cot \frac{\tau}{2})^2 + y^2]}, \\ \frac{by}{\sin^2 \frac{\tau}{2} [(x + \cot \frac{\tau}{2})^2 + y^2]}, \\ \left[\theta + 2 \tan^{-1} \left(\frac{y}{x + \cot \frac{\tau}{2}} \right) \right] \end{pmatrix}. \end{aligned} \quad (14)$$

Certainly the universal cover of $U\mathbb{H}^2$ is diffeomorphic to $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$, say consisting of elements like (x, y, θ) . We may lift the multiplication map

$$\mu : \widetilde{U\mathbb{H}^2} \times \widetilde{U\mathbb{H}^2} \rightarrow \widetilde{U\mathbb{H}^2}$$

given by (14) to a map

$$\tilde{\mu} : \widetilde{U\mathbb{H}^2} \times \widetilde{U\mathbb{H}^2} \rightarrow \widetilde{U\mathbb{H}^2}$$

where we specify that $\tilde{\mu}((0, 1, 0), (0, 1, 0)) = (0, 1, 0)$. The first two coordinates are unchanged, but we must lift the map to the third coordinates. We denote by $\tilde{\mu}_3$ the lifted map to the third coordinate. We see by (14) that $\tilde{\mu}_3 = \tilde{\mu}_3(\tau, x, y, \theta)$. To get a handle on the lift, we need to see when

$$\cos \frac{\theta}{2} \left(x + \cot \frac{\tau}{2} \right) - y \sin \frac{\theta}{2} = 0,$$

i.e.,

$$\cot \frac{\tau}{2} = y \tan \frac{\theta}{2} + x.$$

Note that if $x = 0$ and $y = 1$, then the solutions are $\tau = \pi(2k + 1) - \theta$ for any $k \in \mathbb{Z}$. For general $x \in \mathbb{R}$ and $y \in \mathbb{R}_+$, the solutions are wiggly lines which roughly follow those lines, as the set of all lines is invariant under translation by multiples of 2π in both the up/down and left/right directions (see Figure 1). In particular, we see that on these curves τ is decreasing as a function

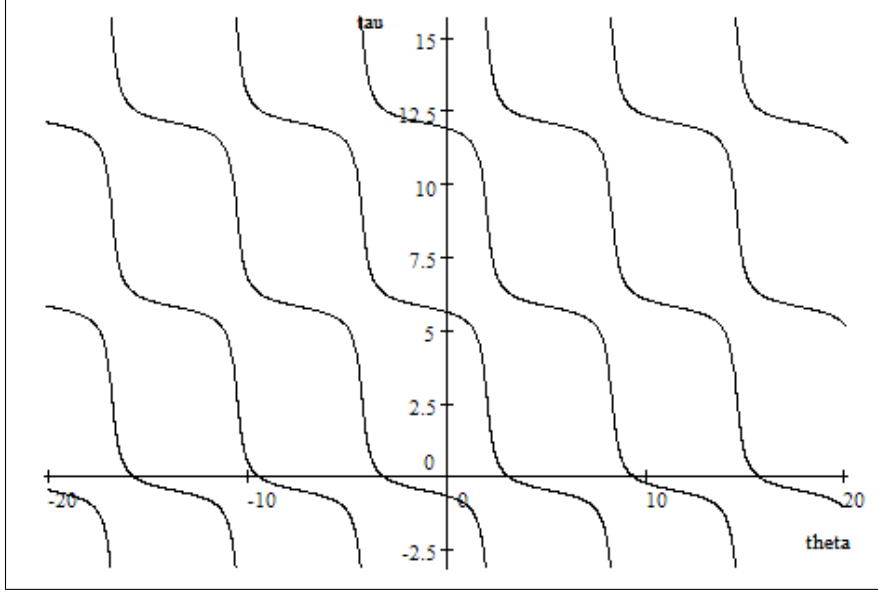


Figure 1: Solutions of $\cot \frac{\tau}{2} = y \tan \frac{\theta}{2} + x$ for $x = -3$ and $y = 2$.

of θ and that the curves are invariant under translations in τ and θ by integer multiples of 2π . There is one curve which intersects $(\theta, \tau) = (\pi, 0)$ and that curve also intersects $(0, \pi - 2 \tan^{-1} x)$. The behavior can be understood by looking at blocks $(\theta, \tau) \in [0, 2\pi] \times [0, 2\pi]$ (in addition to translations of this curve by multiples of 2π in both directions). We see that if θ and τ are positive, then $\tilde{\mu}_3$ is essentially

$$\begin{aligned} \tilde{\mu}_3(\tau, x, y, \theta) &\approx 2 \tan^{-1} \left(\frac{\sin \frac{\theta}{2} (x + \cot \frac{\tau}{2}) + y \cos \frac{\theta}{2}}{\cos \frac{\theta}{2} (x + \cot \frac{\tau}{2}) - y \sin \frac{\theta}{2}} \right) + 2\pi \left\lfloor \frac{\theta + \tau}{2\pi} \right\rfloor \\ &\approx \theta + 2 \tan^{-1} \left(\frac{y}{(x + \cot \frac{\tau}{2})} \right) + 2\pi \left\lfloor \frac{\tau}{2\pi} \right\rfloor \end{aligned} \quad (15)$$

where $\lfloor \cdot \rfloor$ is the floor (greatest integer less than).

Remark 35 Another way to get coordinates on $\widetilde{\text{SL}}(2, \mathbb{R})$ is to first consider the Iwasawa decomposition of $\text{SL}(2, \mathbb{R})$, which says that every matrix in $\text{SL}(2, \mathbb{R})$ can be written as

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} y & x \\ 0 & \frac{1}{y} \end{pmatrix}$$

where $y > 0$. In order to write down the multiplication, one needs to rewrite the product in this form again, and then lift to the universal cover. It turns out that to perform these two operations, the specifics of how x , y , and θ change in this

coordinate chart is much the same as the way they behave in the first coordinate chart we gave. This is because the Iwasawa decomposition of $\mathrm{SL}(2, \mathbb{R})$ acts in an easy way on $U\mathbb{H}^2$, i.e., the rotation $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ fixes i and changes the direction of the vector, and the matrix $\begin{pmatrix} y & x \\ 0 & \frac{1}{y} \end{pmatrix}$ acts by moving i to another point in \mathbb{H}^2 but fixing the direction.

There is a basis of left invariant vector fields given by

$$\begin{aligned} F_1 &= -\frac{\partial}{\partial \theta}, \\ F_2 &= y \cos \theta \frac{\partial}{\partial x} - y \sin \theta \frac{\partial}{\partial y} + \cos \theta \frac{\partial}{\partial \theta}, \\ F_3 &= y \sin \theta \frac{\partial}{\partial x} + y \cos \theta \frac{\partial}{\partial y} + \sin \theta \frac{\partial}{\partial \theta}, \end{aligned}$$

and we can see easily that

$$\begin{aligned} [F_1, F_2] &= F_3, \\ [F_2, F_3] &= -F_1, \\ [F_3, F_1] &= F_2. \end{aligned}$$

The following is a family of left invariant metrics:

$$A \left(d\theta - \frac{1}{y} dx \right)^2 + B \left(\frac{1}{y} \cos \theta dx - \frac{1}{y} \sin \theta dy \right)^2 + C \left(\frac{1}{y} \sin \theta dx + \frac{1}{y} \cos \theta dy \right)^2. \quad (16)$$

If $B = C$, then the metric is

$$g = A \left(d\theta - \frac{1}{y} dx \right)^2 + \frac{B}{y^2} (dx^2 + dy^2)$$

and the metric is a bundle over \mathbb{H}^2 . Note that the isometry group for the general metrics (16) is $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, but if $B = C$, there is an additional part which includes arbitrary translations of θ , so that the isometry group looks like $\widetilde{\mathrm{SL}}(2, \mathbb{R}) \times \mathbb{R}$. (Actually, when $B = C$ this is only the identity component; there are two components due to the isometry $(x, y, \theta) \rightarrow (-x, y, -\theta)$. For more, see [Sc-83].)

The sectional curvatures are:

$$\begin{aligned} K(F_2 \wedge F_3) &= \frac{-3A^2 + B^2 + C^2 - 2AB - 2BC - 2AC}{4ABC}, \\ K(F_3 \wedge F_1) &= \frac{A^2 - 3B^2 + C^2 - 2AB + 2BC + 2AC}{4ABC}, \\ K(F_1 \wedge F_2) &= \frac{A^2 + B^2 - 3C^2 + 2AB + 2BC - 2AC}{4ABC}. \end{aligned}$$

If $B = C$, the sectional curvatures are:

$$\begin{aligned} K(F_2 \wedge F_3) &= \frac{-3A - 4B}{4B^2}, \\ K(F_3 \wedge F_1) &= \frac{A}{4B^2}, \\ K(F_1 \wedge F_2) &= \frac{A}{4B^2}. \end{aligned}$$

Note that we have taken F_1, F_2, F_3 which are one half that in [KM-01] and [CNS-07], so our A, B, C are all one fourth the corresponding coefficients in those papers.

4.3.1 Ricci Flow

Under Ricci flow, from [KM-01] we have that A goes to a constant, B and C are like $2t$ and

$$|B - C| \leq E_1 e^{-E_2 t},$$

for positive constants E_1 and E_2 . We see that the sectional curvatures can be written as

$$\begin{aligned} K(F_2 \wedge F_3) &= \frac{(B - C)^2 - A(3A + 2B + 2C)}{4ABC}, \\ K(F_3 \wedge F_1) &= \frac{(A - (B - C))^2 - 4B(B - C)}{4ABC}, \\ K(F_1 \wedge F_2) &= \frac{(A + (B - C))^2 + 4C(B - C)}{4ABC}, \end{aligned}$$

and so

$$\begin{aligned} K(F_2 \wedge F_3) &\sim \frac{E_3}{t}, \\ K(F_3 \wedge F_1) &\sim \frac{E_4}{t^2}, \\ K(F_1 \wedge F_2) &\sim \frac{E_5}{t^2} \end{aligned}$$

for large t , where E_3, E_4, E_5 are constants. Thus the solution is Type III.

We may do a Type III rescaling and pull back by the diffeomorphism

$$\phi_s(x, y, \theta) = \phi_s\left(x, y, s^{1/2}\theta\right)$$

to get the limit

$$\lim_{s \rightarrow \infty} \frac{1}{s} \phi_s^* g(st) = d\theta^2 + \frac{2t}{y^2} (dx^2 + dy^2),$$

which is an expanding soliton on $\mathbb{H}^2 \times \mathbb{R}$.

Let's look at the evolution of the action of the isometry group. We see that

$$\phi_s^{-1}(a, b, \tau) \phi_s(x, y, \theta) = \begin{pmatrix} a + \frac{b(x \cos \tau + \frac{1}{2}(x^2 + y^2 - 1) \sin \tau)}{\sin^2 \frac{\tau}{2} [(x + \cot \frac{\tau}{2})^2 + y^2]}, \\ \frac{by}{\sin^2 \frac{\tau}{2} [(x + \cot \frac{\tau}{2})^2 + y^2]}, \\ s^{-1/2} \tilde{\mu}_3(\tau, x, y, s^{1/2} \theta) \end{pmatrix}.$$

We note that for large s ,

$$s^{-1/2} \tilde{\mu}_3(\tau, x, y, s^{1/2} \theta) \approx \theta + s^{-1/2} \tan^{-1} \left(\frac{y}{x + \cot \frac{\tau}{2}} \right) + 2\pi s^{-1/2} \left\lfloor \frac{\tau}{2\pi} \right\rfloor.$$

Thus if $\tau(s)$ is bounded, then as $s \rightarrow \infty$ we get elements that look like:

$$\gamma_{(a,b,\tau)}(x, y, \theta) = \begin{pmatrix} a + \frac{b(x \cos \tau + \frac{1}{2}(x^2 + y^2 - 1) \sin \tau)}{\sin^2 \frac{\tau}{2} [(x + \cot \frac{\tau}{2})^2 + y^2]}, \\ \frac{by}{\sin^2 \frac{\tau}{2} [(x + \cot \frac{\tau}{2})^2 + y^2]}, \\ \theta \end{pmatrix}.$$

If $\tau(s) \rightarrow \infty$, then we may have other elements. For the first two components to make sense, we need to take a sequence where $\tau_i = \tau_0 + 2\pi k_i$ for $k_i \in \mathbb{Z}$. If we take such a sequence of τ_i and a sequence of $s_i \rightarrow \infty$ such that

$$\lim_{i \rightarrow \infty} 2\pi s_i^{-1/2} \left\lfloor \frac{\tau_i}{2\pi} \right\rfloor = u$$

for $u \in \mathbb{R}$, we see that we can get as a limit elements that look like

$$\gamma_{(a,b,\tau,u)}(x, y, \theta) = \begin{pmatrix} a + \frac{b(x \cos \tau + \frac{1}{2}(x^2 + y^2 - 1) \sin \tau)}{\sin^2 \frac{\tau}{2} [(x + \cot \frac{\tau}{2})^2 + y^2]}, \\ \frac{by}{\sin^2 \frac{\tau}{2} [(x + \cot \frac{\tau}{2})^2 + y^2]}, \\ \theta + u \end{pmatrix}.$$

In particular, if $\tau_0 = 0$ and $a = 0, b = 1$ then we get the translations

$$\gamma_{(0,1,0,u)}(x, y, \theta) = \begin{pmatrix} x, \\ y, \\ \theta + u \end{pmatrix}$$

for any $u \in \mathbb{R}$. Note that for simplicity of the formula in (15) we assumed that τ and θ are positive, but one can also do the general case with a careful analysis of the lifted map $\tilde{\mu}_3$.

Thus the isometry group converges the standard action of $\text{PSL}(2, \mathbb{R})$ on \mathbb{H}^2 in the first two coordinates and a continuous action in the last coordinate, i.e., to $\text{PSL}(2, \mathbb{R}) \times \mathbb{R}$.

We may consider a compact quotient $\widetilde{\text{SL}}(2, \mathbb{R})/\Gamma$ where Γ acts properly discontinuously (and freely if the quotient is a manifold). A properly discontinuous group action may still contain the \mathbb{R} action since τ may grow without bound, and so the limit group will still be continuous. Thus the orbit space of the groupoid will be a two-dimensional quotient of $\mathbb{H}^2 \times \mathbb{R}$.

4.3.2 Cross Curvature Flow

The negative cross curvature flow exhibits two different types of behavior, so we divide it into two cases.

Case 1: $B_0 = C_0$. In this case, the solution exists for all $t \in [0, \infty)$ and $B(t) = C(t)$ for all t . The solutions are

$$A \sim A_\infty, \quad B = C \sim \left(\frac{3}{2} A_\infty t \right)^{1/3},$$

where $A_\infty > 0$ is the limit of A , which decreases monotonically to it. The sectional curvatures are

$$\begin{aligned} K(F_2 \wedge F_3) &= \frac{-3A - 4B}{4B^2} \sim -\frac{1}{\left(\frac{3}{2} A_\infty t\right)^{1/3}}, \\ K(F_3 \wedge F_1) &= \frac{A}{4B^2} \sim \frac{A_\infty}{4 \left(\frac{3}{2} A_\infty t\right)^{2/3}}, \\ K(F_1 \wedge F_2) &= \frac{A}{4B^2} \sim \frac{A_\infty}{4 \left(\frac{3}{2} A_\infty t\right)^{2/3}}. \end{aligned}$$

We see that this is a Type IIb solution since one sectional curvature does not decay faster than $t^{-1/2}$. We may take the geometric limit

$$\begin{aligned} & \lim_{s \rightarrow \infty} s^{-1/3} \phi_s^* g \left(s^{2/3} (t-1) + s \right) \\ &= \lim_{s \rightarrow \infty} \left[s^{-1/3} A_\infty \left(s^{1/6} d\theta - \frac{1}{y} dx \right)^2 + s^{-1/3} \left(\frac{3}{2} A_\infty \right)^{1/3} \left(s^{2/3} (t-1) + s \right)^{1/3} \left(\frac{1}{y^2} dx^2 + \frac{1}{y^2} dy^2 \right) \right] \\ &= A_\infty d\theta^2 + \left(\frac{3}{2} A_\infty \right)^{1/3} \frac{1}{y^2} (dx^2 + dy^2) \end{aligned}$$

where

$$\phi_s(x, y, \theta) = \left(x, y, s^{1/6} \theta \right).$$

This is the solution $\mathbb{H}^2 \times \mathbb{R}$, which is a fixed point of the flow. (In fact, since the cross curvature tensor consists of products of two of the sectional curvatures, any product metric is a fixed point.) The compact quotients behave the same way as the Ricci flow case.

Case 2: $B_0 > C_0$ In this case, [CNS-07] shows that

$$A, B \sim E (T_0 - t)^{-1/2}, \quad C \sim 2\sqrt{T_0 - t},$$

for some singularity time $T_0 > 1$ and that

$$\begin{aligned}\lim_{t \rightarrow T_0} C K (F_2 \wedge F_3) &= -1, \\ \lim_{t \rightarrow T_0} C K (F_3 \wedge F_1) &= -1, \\ \lim_{t \rightarrow T_0} C K (F_1 \wedge F_3) &= 1.\end{aligned}$$

Thus the sectional curvatures blow up like

$$\frac{1}{2\sqrt{T_0 - t}},$$

and the singularity is Type I. We look at the Type I limit:

$$\begin{aligned}& \lim_{s \rightarrow 0} s^{-1/2} \phi_s^* g (T_0 - st) \\ &= \lim_{s \rightarrow 0} \left[E s^{-1} t^{-1/2} \left(s^{1/2} d\theta - \frac{1}{y} s^{1/2} dx \right)^2 + E s^{-1} t^{-1/2} \left(\frac{1}{y} \cos(s^{1/2} \theta) s^{1/2} dx - \frac{1}{y} \sin(s^{1/2} \theta) dy \right)^2 \right. \\ & \quad \left. + 2t^{1/2} \left(\frac{1}{y} \sin(s^{1/2} \theta) s^{1/2} dx + \frac{1}{y} \cos(s^{1/2} \theta) dy \right)^2 \right] \\ &= E t^{-1/2} \left(d\theta - \frac{1}{y} dx \right)^2 + E t^{-1/2} \left(\frac{1}{y} dx - \frac{\theta}{y} dy \right)^2 + 2t^{1/2} \left(\frac{1}{y} dy \right)^2,\end{aligned}$$

where

$$\phi_s(x, y, \theta) = (s^{1/2} x, y, s^{1/2} \theta).$$

We claim that this is actually the XC soliton on Sol. We can see this if we pull back by

$$\psi(x, y, \theta) = (e^y x, e^y, \theta + x) = (\tilde{x}, \tilde{y}, \tilde{\theta})$$

to get

$$\begin{aligned}& E t^{-1/2} \left(d\tilde{\theta} - \frac{1}{\tilde{y}} d\tilde{x} \right)^2 + E t^{-1/2} \left(\frac{1}{\tilde{y}} d\tilde{x} - \frac{\tilde{\theta}}{\tilde{y}} d\tilde{y} \right)^2 + 2t^{1/2} \left(\frac{1}{\tilde{y}} d\tilde{y} \right)^2 \\ &= E t^{-1/2} (d\theta - x dy)^2 + E t^{-1/2} (dx - \theta dy)^2 + 2t^{1/2} dy^2,\end{aligned}$$

which can easily be transformed by rescaling coordinates to the metric (11).

We let

$$\begin{aligned}\psi_s(x, y, \theta) &= \phi_s \circ \psi(x, y, \theta) = (s^{1/2} e^y x, e^y, s^{1/2} (\theta + x)) \\ \psi_s^{-1}(x, y, \theta) &= \left(s^{-1/2} \frac{x}{y}, \log y, s^{-1/2} \left(\theta - \frac{x}{y} \right) \right)\end{aligned}$$

Now consider what happens to the group action in the limit. We compute

$$\begin{aligned}
& \psi_s^{-1}(a, b, \tau) \psi_s(x, y, \theta) \\
&= \psi_s^{-1} \left(\begin{array}{c} a + \frac{b(s^{1/2}e^y x \cos \tau + \frac{1}{2}(se^{2y}x^2 + e^{2y} - 1) \sin \tau)}{\sin^2 \frac{\tau}{2} [(s^{1/2}e^y x + \cot \frac{\tau}{2})^2 + y^2]}, \\ \frac{be^y}{\sin^2 \frac{\tau}{2} [(s^{1/2}e^y x + \cot \frac{\tau}{2})^2 + e^{2y}]}, \\ \tilde{\mu}_3(\tau, s^{1/2}e^y x, e^y, s^{1/2}(\theta + x)) \end{array} \right) \\
&= \begin{pmatrix} \alpha, \\ \beta, \\ \gamma, \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
\alpha_s(a, b, \tau, x, y) &= s^{-1/2} \frac{a}{b} e^y \left(\left(s^{1/2} x \sin \frac{\tau}{2} + e^{-y} \cos \frac{\tau}{2} \right)^2 + \sin^2 \frac{\tau}{2} \right) \\
&\quad + \left(x \cos \tau + \frac{1}{2} \left(s^{1/2} e^y x^2 + s^{-1/2} e^y - s^{-1/2} e^{-y} \right) \sin \tau \right), \\
\beta_s(b, \tau, x, y) &= y + \log b - \log \left(\left(s^{1/2} e^y x \sin \frac{\tau}{2} + \cos \frac{\tau}{2} \right)^2 + e^{2y} \sin^2 \frac{\tau}{2} \right),
\end{aligned}$$

and

$$\gamma_s(a, b, \tau, x, y, \theta) = s^{-1/2} \tilde{\mu}_3(\tau, s^{1/2}e^y x, e^y, s^{1/2}(\theta + x)) - \alpha_s(a, b, \tau, x, y).$$

We see immediately that since α_s cannot become unbounded as $s \rightarrow 0$, for γ_s to not be unbounded, we need

$$s^{-1/2} \tilde{\mu}_3(\tau, s^{1/2}e^y x, e^y, s^{1/2}(\theta + x))$$

to stay bounded. Since the last term for $\tilde{\mu}_3$ goes to zero, this is a restriction on our choice of $\tau(s)$. In particular, we need that τ stays relatively close to zero. Knowing this, we can look more precisely at the formula for this term, which, for positive τ , is

$$\begin{aligned}
& s^{-1/2} \tilde{\mu}_3(\tau, s^{1/2}e^y x, e^y, s^{1/2}(\theta + x)) \\
& \approx \theta + x + 2s^{-1/2} \tan^{-1} \left(\frac{e^y \tan \frac{\tau}{2}}{(s^{1/2}e^y x \tan \frac{\tau}{2} + 1)} \right) + 2\pi s^{-1/2} \left\lfloor \frac{\tau}{2\pi} \right\rfloor.
\end{aligned}$$

Since τ is close to zero, the last term is always zero, and for the second to last term to not go to infinity, we need

$$\lim_{s \rightarrow 0} s^{-1/2} \tau = 2u,$$

for some $u \in \mathbb{R}$, in which case we get

$$\lim_{s \rightarrow 0} \gamma_s(a, b, \tau, x, y, \theta) = \theta + x + 2e^y u - \lim_{s \rightarrow 0} \alpha_s(a, b, \tau, x, y).$$

Considering β_s and the fact that $\tau(s) \rightarrow 0$ as $s \rightarrow 0$, we see that b must converge to a positive number as $s \rightarrow 0$, so we might as well assume $b(s) \rightarrow e^d$. Thus

$$\lim_{s \rightarrow 0} \beta_s(b, \tau, x, y) = y + d.$$

Finally, if we have $v \in \mathbb{R}$ such that

$$\lim_{s \rightarrow 0} s^{-1/2} \frac{a(s)}{b(s)} = u + v,$$

we have

$$\begin{aligned} \lim_{s \rightarrow 0} \alpha_s(a, b, \tau, x, y) &= x + e^{-y}v + (e^y - e^{-y})u \\ &= x + e^y u + e^{-y}v. \end{aligned}$$

Thus,

$$\lim_{s \rightarrow 0} \gamma_s(a, b, \tau, x, y, \theta) = \theta + e^y u - e^{-y}v.$$

The limit group actions consist of maps

$$\gamma_{(u,v,d)}(x, y, \theta) = (x + e^y u + e^{-y}v, y + d, \theta + e^y u - e^{-y}v).$$

Note that this has the form of the isometries described by (12) modulo the change of coordinates defined by scaling y .

Now, if we begin with a groupoid representing a compact manifold quotient of $\widetilde{\text{SL}}(2, \mathbb{R})$, the group which determines the arrows in the groupoid must act properly discontinuously. We see that this implies that u and v are zero, and d takes discrete values. Thus the limit is a noncompact quotient of Sol with no collapsing.

4.4 $\widetilde{\text{Isom}}(\mathbb{E}^2)$

The group $\text{Isom}(\mathbb{E}^2)$ consists of group elements

$$(x^1, x^2) \rightarrow A_\theta \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix},$$

where A_θ is a rotation by angle θ . Thus the universal cover is diffeomorphic to \mathbb{R}^3 and has coordinates (x, y, θ) . The group multiplication is

$$(a, b, \tau)(x, y, \theta) = (x \cos \tau + y \sin \tau + a, -x \sin \tau + y \cos \tau + b, \theta + \tau).$$

This group has a left invariant frame

$$F_1 = \sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y}, \quad F_2 = \cos \theta \frac{\partial}{\partial x} - \sin \theta \frac{\partial}{\partial y}, \quad F_3 = \frac{\partial}{\partial \theta}$$

with

$$\begin{aligned} [F_2, F_3] &= F_1 \\ [F_3, F_1] &= F_2 \end{aligned}$$

the only nonzero brackets. Thus the following are left invariant metrics:

$$g = A (\sin \theta dx + \cos \theta dy)^2 + B (\cos \theta dx - \sin \theta dy)^2 + C d\theta^2.$$

We note that if $A = B$, then the metric is Euclidean. It is clear by changing coordinates by scaling x and y that this is really a two parameter family of metrics up to diffeomorphism.

The sectional curvatures for these metrics are

$$\begin{aligned} K(F_2 \wedge F_3) &= \frac{(B - A)(B + 3A)}{4ABC}, \\ K(F_3 \wedge F_1) &= \frac{(A - B)(A + 3B)}{4ABC}, \\ K(F_1 \wedge F_2) &= \frac{(A - B)^2}{4ABC}. \end{aligned}$$

Note that this F_3 is one half that used in [KM-01] and [CNS-07], so our C is 1/4 the corresponding coefficient in those papers.

4.4.1 Ricci Flow

From [KM-01], we see that the solution to the Ricci flow looks like

$$\begin{aligned} A, B &\sim E_1, \\ C &\sim E_2, \end{aligned}$$

where $E_1 = \sqrt{A_0 B_0}$ and $E_2 = \frac{C_0}{2} \left(\sqrt{\frac{A_0}{B_0}} + \sqrt{\frac{B_0}{A_0}} \right)$. From the work in [KM-01], we easily see that

$$\frac{d}{dt} (A - B) = - (A - B) \frac{(A + B)}{C} \left(\frac{1}{B} + \frac{1}{A} \right),$$

and so

$$A - B \sim E_4 e^{-E_3 t},$$

where

$$\begin{aligned} E_3 &= \frac{4}{E_2}, \\ E_4 &= (A_0 - B_0). \end{aligned}$$

The sectional curvatures are

$$\begin{aligned} K(F_2 \wedge F_3) &\sim -\frac{4E_4 E_1}{E_1^2 E_2} e^{-E_3 t}, \\ K(F_3 \wedge F_1) &\sim \frac{4E_4 E_1}{E_1^2 E_2} e^{-E_3 t}, \\ K(F_1 \wedge F_2) &\sim \frac{E_4^2}{E_1^2 E_2} e^{-2E_3 t}. \end{aligned}$$

This is a Type III solution. It is clear that the Type III limit is Euclidean space since

$$g = A(dx^2 + dy^2) + (B - A)(\cos \theta dx - \sin \theta dy)^2 + Cd\theta^2$$

and so

$$\begin{aligned} g_\infty(t) &= \lim_{s \rightarrow \infty} s^{-1} \phi_s^* g(st) \\ &= \lim_{s \rightarrow \infty} s^{-1} \left[E_1 s (dx^2 + dy^2) + E_4 e^{-E_3 s t} s \left(\cos s^{1/2} \theta dx - \sin s^{1/2} \theta dy \right)^2 + E_2 s d\theta^2 \right] \\ &= E_1 (dx^2 + dy^2) + E_2 d\theta^2, \end{aligned}$$

where

$$\phi_s(x, y, \theta) = \left(s^{1/2} x, s^{1/2} y, s^{1/2} \theta \right).$$

One might try to construct a different geometric limit by choosing a different rescaling, for instance the following:

$$\begin{aligned} g_s(t) &= e^{-E_3 s} \psi_s^* g(e^{E_3 s}(t-1) + s) \\ &= e^{-E_3 s} E_1 e^{E_3 s} (dx^2 + dy^2) \\ &\quad + e^{-E_3 s} E_4 e^{-E_3(e^{E_3 s}(t-1)+s)} e^{E_3 s} \left(\cos(e^{E_3 s/2} \theta) dx - \sin(e^{E_3 s/2} \theta) dy \right)^2 \\ &\quad + e^{-E_3 s} E_2 e^{E_3 s} d\theta^2 \\ &= E_1 (dx^2 + dy^2) + E_4 e^{-E_3(e^{E_3 s}(t-1)+s)} \left(\cos(e^{E_3 s/2} \theta) dx - \sin(e^{E_3 s/2} \theta) dy \right)^2 \\ &\quad + E_2 d\theta^2 \end{aligned}$$

where

$$\psi_s(x, y, \theta) = \left(e^{E_3 s/2} x, e^{E_3 s/2} y, e^{E_3 s/2} \theta \right).$$

Notice that as $s \rightarrow \infty$, this also converges to Euclidean space.

Under the Type III limit, the limit of the group actions is

$$\begin{aligned} &\lim_{s \rightarrow \infty} \phi_s^{-1} [\gamma_{(a,b,\tau)} \phi_s(x, y, \theta)] \\ &= \lim_{s \rightarrow \infty} \left(x \cos \tau + y \sin \tau + s^{-1/2} a, -x \sin \tau + y \cos \tau + s^{-1/2} b, \theta + s^{-1/2} \tau \right) \\ &= (x \cos \tau + y \sin \tau + u, -x \sin \tau + y \cos \tau + v, \theta + w) \end{aligned}$$

if we choose $a(s)$ and $b(s)$ so that

$$\begin{aligned} \lim_{s \rightarrow \infty} s^{-1/2} a(s) &= u, \\ \lim_{s \rightarrow \infty} s^{-1/2} b(s) &= v, \\ \lim_{s \rightarrow \infty} s^{-1/2} \tau(s) &= w, \end{aligned}$$

for any $u, v, w \in \mathbb{R}$ and we choose $\tau(s)$ so that it is growing in multiples of 2π (so $\cos \tau$ and $\sin \tau$ still make sense). Thus the limit group is $\text{Isom } \mathbb{E}^2 \times \mathbb{R}$. For a compact quotient, we may still find a, b, τ that grow as desired, so we still get the whole group in the limit. Thus the orbit space of the Type III limit is a point.

4.4.2 Cross Curvature Flow

The solution to -XCF is

$$\begin{aligned} A &\sim E_1, \\ B &\sim E_1, \\ C &\sim \frac{2E_2}{E_1} \sqrt{6} t^{1/3}, \end{aligned}$$

with

$$A - B \sim E_2 t^{-1/6}.$$

Thus the sectional curvatures satisfy

$$\begin{aligned} K(F_2 \wedge F_3) &\sim -\frac{1}{2\sqrt{6}t^{1/2}}, \\ K(F_3 \wedge F_1) &\sim \frac{1}{2\sqrt{6}t^{1/2}}, \\ K(F_1 \wedge F_2) &\sim \frac{E_2}{8E_1\sqrt{6}t^{2/3}}, \end{aligned}$$

and the solution is Type III. We may compute the Type III limit of the rescaled solutions

$$\begin{aligned} g_s(t) &= s^{-1/2} \phi_s^* g(st) \\ &= s^{-1/2} E_1 s^{1/2} (dx^2 + dy^2) \\ &\quad + s^{-1/2} E_2 s^{-1/6} t^{-1/6} s^{1/2} \left(\cos s^{1/12} \theta \, dx - \sin s^{1/12} \theta \, dy \right)^2 \\ &\quad + s^{-1/2} \frac{2E_2}{E_1} \sqrt{6} s^{1/3} t^{1/3} s^{1/6} d\theta^2, \end{aligned}$$

where

$$\phi_s(x, y, \theta) = \left(s^{1/4} x, s^{1/4} y, s^{1/12} \theta \right).$$

The limit as $s \rightarrow \infty$ is

$$g_\infty(t) = E_1 (dx^2 + dy^2) + \frac{2E_2}{E_1} \sqrt{6} t^{1/3} \left(\frac{1}{2} d\theta \right)^2.$$

This is obviously Euclidean space.

Geometry	Soliton	Sing. Type	Geometry Limit	Collapsing limit (dim)	Compact limit
Nil	Yes	III	Nil	Yes (0)	Yes
Sol	Yes	III	Sol	Yes (1)	Yes
$\widetilde{\text{SL}}(2, \mathbb{R})$	No	III	$\mathbb{H}^2 \times \mathbb{R}$	Yes (2)	Yes
$\widetilde{\text{Isom}}(\mathbb{E}^2)$	Yes	III	\mathbb{E}^3	Yes (0)	Yes

Figure 2: Summary of limits of Ricci flow

The pulled back group actions look like

$$\begin{aligned} & \phi_s^{-1} \circ \gamma_{(a,b,\tau)} \circ \phi_s(x, y, \theta) \\ &= \left(x \cos \tau + y \sin \tau + s^{-1/4}a, -x \sin \tau + y \cos \tau + s^{-1/4}b, \theta + s^{-1/12}\tau \right). \end{aligned}$$

So in the limit, we may take

$$\begin{aligned} \lim_{s \rightarrow \infty} s^{-1/4}a(s) &= u \\ \lim_{s \rightarrow \infty} s^{-1/4}b(s) &= v \\ \lim_{s \rightarrow \infty} s^{-1/12}\tau(s) &= w \end{aligned}$$

to get group actions

$$\gamma_{(\tau,u,v,w)}(x, y, \theta) = (x \cos \tau + y \sin \tau + u, -x \sin \tau + y \cos \tau + v, \theta + w)$$

if we take the limit so that $\tau(s)$ is growing only by multiples of 2π so that $\sin \tau$ and $\cos \tau$ still make sense. Note that even if the original groupoid comes from a compact quotient, we still get the entirety of the group since we can let a, b, τ grow (these elements exist in the lattice). Thus the limit has an orbit space of a point.

5 Summary

We may summarize the results about the limits of compact quotients of homogeneous geometries as follows. See also the tables in Figures 2 and 3 for a summary. In the tables, the geometry limit is the limit of the universal covers (or limit of $G^{(0)}$) and the dimension (dim) is the dimension of the orbit space.

Theorem 36 (Lott [Lo-05]) *The solutions of Ricci flow on Nil, Sol, $\widetilde{\text{SL}}(2, \mathbb{R})$, and $\widetilde{\text{Isom}}(\mathbb{E}^2)$ are all of Type III. There are soliton solutions on Nil, Sol, and $\widetilde{\text{Isom}}(\mathbb{E}^2)$ (this soliton is \mathbb{E}^3 , three-dimensional Euclidean space) and the Type III limits converge to these geometries. The Type III limit of $\widetilde{\text{SL}}(2, \mathbb{R})$ is $\mathbb{H}^2 \times \mathbb{R}$. Compact homogeneous manifolds with these geometries all collapse and stay compact.*

Geometry	Soliton	Sing. Type	Geometry Limit	Collapsing limit (dim)	Compact limit
Nil	Yes	III	Nil	Yes (0)	Yes
Sol	Yes	I	Sol	No	No
$\widetilde{\text{SL}}(2, \mathbb{R})$, $B = C$	No	IIb	$\mathbb{H}^2 \times \mathbb{R}$	Yes (2)	Yes
$\widetilde{\text{SL}}(2, \mathbb{R})$, $B \neq C$	No	I	Sol	No	No
$\widetilde{\text{Isom}}(\mathbb{E}^2)$	Yes	III	\mathbb{E}^3	Yes (0)	Yes

Figure 3: Summary of limits of negative cross curvature flow

Theorem 37 *The solutions of cross curvature flow are as follows:*

- Nil admits a XC soliton metric. Its solution is Type III and compact manifolds modeled on Nil converge to compact, collapsed quotients of Nil in the Type III limit.
- Sol admits a XC soliton metric. Its solution is Type IIa and compact manifolds modeled on Sol converge to noncollapsed, noncompact quotients of Sol.
- $\widetilde{\text{SL}}(2, \mathbb{R})$ does not seem to admit a soliton metric. If $B = C$ (i.e., if the metric is a Riemannian submersion over \mathbb{H}^2), then the singularity is Type IIb and the geometric limits of compact manifolds modeled on this type of $\widetilde{\text{SL}}(2, \mathbb{R})$ are collapsed, compact quotients of $\mathbb{H}^2 \times \mathbb{R}$. If $B \neq C$, then the singularity is Type I and the type I limits of compact manifolds modeled on this type of $\widetilde{\text{SL}}(2, \mathbb{R})$ converge to noncollapsed, noncompact quotients of Sol.
- $\widetilde{\text{Isom}}(\mathbb{E}^2)$ admits a soliton metric which is isometric to \mathbb{E}^3 . The singularity is Type III and the Type III limit of compact manifolds modeled on $\widetilde{\text{Isom}}(\mathbb{E}^2)$ consist of compact, collapsed quotients of \mathbb{E}^3 .

References

- [BD-05] P. Baird and L. Danielo. Three-dimensional Ricci solitons which project to surfaces. Preprint at arXiv:math/0510313v1 [math.DG].
- [BR-99] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319. Springer-Verlag, Berlin, 1999.
- [Buc-06] J. A. Buckland. Short-time existence of solutions to the cross curvature flow on 3-manifolds. Proc. Amer. Math. Soc. 134 (2006), no. 6, 1803–1807.

- [BK-81] P. Buser and H. Karcher. *Gromov's almost flat manifolds*. Astérisque, 81. Société Mathématique de France, Paris, 1981.
- [CNS-07] X. Cao, Y. Ni, and L. Saloff-Coste. Cross curvature flow on locally homogeneous three manifolds (I). Preprint at arXiv:0708.1922v1 [math.DG].
- [CZ-06] H.-D. Cao and X.-P. Zhu. A complete proof of the Poincaré and geometrization conjectures—application of the Hamilton-Perelman theory of the Ricci flow. *Asian J. Math.* 10 (2006), no. 2, 165–492.
- [CCG+07] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, L. Ni. *The Ricci flow: techniques and applications. Part I. Geometric aspects*. Mathematical Surveys and Monographs, 135. American Mathematical Society, Providence, RI, 2007.
- [CH-04] B. Chow and R. S. Hamilton. The cross curvature flow of 3-manifolds with negative sectional curvature. *Turkish J. Math.* 28 (2004), no. 1, 1–10.
- [CK-04] B. Chow and D. Knopf. *The Ricci flow: an introduction*. Mathematical Surveys and Monographs, 110. American Mathematical Society, Providence, RI, 2004. xii+325 pp.
- [CLN-06] B. Chow, P. Lu, and L. Ni. *Hamilton's Ricci flow*. Graduate Studies in Mathematics, 77. American Mathematical Society, Providence, RI; Science Press, New York, 2006. xxxvi+608 pp.
- [Fu-88] K. Fukaya. A boundary of the set of the Riemannian manifolds with bounded curvatures and diameters. *J. Differential Geom.* 28 (1988), no. 1, 1–21.
- [GGHR-89] E. Gallego, L. Gualandri, G. Hector, and A. Reventós. Groupoïdes riemanniens. (French) [Riemannian groupoids] *Publ. Mat.* 33 (1989), no. 3, 417–422.
- [G-03] D. Glickenstein. Precompactness of solutions to the Ricci flow in the absence of injectivity radius estimates. *Geom. Topol.* 7 (2003), 487–510 (electronic).
- [Gr-78] M. Gromov. Almost flat manifolds. *J. Differential Geom.* 13 (1978), no. 2, 231–241.
- [GIK-06] C. Guenther, J. Isenberg, and D. Knopf. Linear stability of homogeneous Ricci solitons. *Int. Math. Res. Not.* 2006, Art. ID 96253, 30 pp.
- [Hae-88] A. Haefliger. Leaf closures in Riemannian foliations. *A fête of topology*, 3–32, Academic Press, Boston, MA, 1988.

- [Ham-83] R. S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Differential Geom.* 17 (1982), no. 2, 255–306.
- [Ham-95a] R. S. Hamilton. The formation of singularities in the Ricci flow. *Surveys in differential geometry*, Vol. II (Cambridge, MA, 1993), 7–136, Int. Press, Cambridge, MA, 1995.
- [Ham-95b] R. S. Hamilton. A compactness property for solutions of the Ricci flow. *Amer. J. Math.* 117 (1995), no. 3, 545–572.
- [IJ-92] J. Isenberg and M. Jackson. Ricci flow of locally homogeneous geometries on closed manifolds. *J. Differential Geom.* 35 (1992), no. 3, 723–741.
- [KL-06] B. Kleiner and J. Lott. Notes on Perelman’s papers. Preprint at arXiv:math/0605667v2 [math.DG].
- [KM-01] D. Knopf and K. McLeod. Quasi-convergence of model geometries under the Ricci flow. *Comm. Anal. Geom.* 9 (2001), no. 4, 879–919.
- [Lo-05] J. Lott. On the long-time behavior of type-III Ricci flow solutions. *Math. Ann.* 339 (2007), 627–666.
- [Lu-01] P. Lu. A compactness property for solutions of the Ricci flow on orbifolds. *Amer. J. Math.* 123 (2001), no. 6, 1103–1134.
- [Ma-05] K. C. H. Mackenzie. *General theory of Lie groupoids and Lie algebroids*. London Mathematical Society Lecture Note Series, 213. Cambridge University Press, Cambridge, 2005. xxxviii+501 pp.
- [MM-03] I. Moerdijk and J. Mrčun. *Introduction to foliations and Lie groupoids*. Cambridge Studies in Advanced Mathematics, 91. Cambridge University Press, Cambridge, 2003. x+173 pp.
- [MT-06] J. Morgan and G. Tian. Ricci flow and the Poincare conjecture. Preprint at arXiv:math/0607607v2 [math.DG].
- [P-02] G. Perelman. The entropy formula for the Ricci flow and its geometric applications. Preprint at arXiv:math/0211159v1 [math.DG].
- [P-03] G. Perelman. Ricci flow with surgery on three-manifolds. Preprint at arXiv:math/0303109v1 [math.DG].
- [Sc-83] P. Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.* 15 (1983), no. 5, 401–487.
- [Sa-88] É. Salem. Une généralisation du théorème de Myers-Steenrod aux pseudogroupes d’isométries. (French) [A generalization of the Myers-Steenrod theorem to pseudogroups of isometries] *Ann. Inst. Fourier (Grenoble)* 38 (1988), no. 2, 185–200.